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SPT 2001

Symmetry and Perturbation Theory

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Cala Gonone, Sardinia, Italy 6 – 13 May 2001

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SYMMETRY AND PERTURBATION THEORY
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Preface

The third conference on *Symmetry and Perturbation Theory* (**SPT2001**) took place in Cala Gonone, a small village on the beautiful eastern coast of Sardinia, on 6-13 May 2001. This followed the conferences of the same title held in Torino[1] in december 1996 and in Roma[2] in december 1998.

The conference was attended by over 50 mathematicians, physicists and chemists, and was a nice occasion to have interdisciplinary discussion involving rather different communities; we hope that the reader of these proceedings will find within this volume some remnant of the relaxed and fruitful atmosphere we enjoyed in Cala Gonone, and we trust he/she will find plenty of useful information on the advancement of research in this field, or better said in the different fields at whose crossroads symmetry and perturbation theory sit.

In order to respect the interdisciplinary character of the conference, we avoided to separate the papers into specialized sessions, and just collected them in alphabetical order (by author's name).

We also give, together with the conference program and the list of participants, the list of papers appeared in the proceedings of previous SPT conferences.

In the course of the conference we had a special session devoted to Louis Michel – who died on 30 December 1999 – and his influence on the subject of the conference, organized by his collaborator and friend Boris Zhilinskii. This session has seen, after a speech by Boris on Louis' life and work, the talks of Yuri Gufan, James Montaldi, Dimitrii Sadovskii, and Joshua Zak. On the one hand, it would have been natural to put these talks in a special section of these proceedings; but on the other hand, a cursory look at the table of contents will show to anybody slightly familiar with the work of Louis that it would be very reductive to confine his influence to this special session. The words written by Boris on "Symmetry, Perturbation Theory, and Louis Michel" suitably close this volume stressing the influence of Louis in the field.

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Acknowledgements

We would like to stress that we asked our authors a serious effort to have the proceedings ready within less than three months from the conference; we would like to thank them again here for having responded positively to this requirement.

There are also, well sure, a number of individuals and institutions whose help was crucial for the success of the conference.

We would like first of all to thank all those being part of the Scientific Committee of SPT2001 for their constant advice and help. This was made of: Dario Bambusi (Milano), Pascal Chossat (Nice), Giampaolo Cicogna (Pisa), Antonio Degasperis (Roma), Giuseppe Gaeta (Roma and Milano), Jeroen Lamb (London), Giuseppe Marmo (Napoli), Mark Roberts (Warwick and Surrey), Gianfranco Sartori (Padova), Ferdinand Verhulst (Utrecht), Sebastian Walcher (Münich), and Boris Zhilinskii (Dunquerque).

A conference gathering different communities is stimulating, but presents a problem of different backgrounds; to overcome this we asked to a number of people to write "tutorial papers" on some selected topic (these are being published elsewhere[3]). We would like to warmly thank them, and even more those who were in the end unable to attend the conference, for their help.

The *"Pro-Loco"* of the city of Dorgali (in whose territory Cala Gonone lies) was very helpful whenever we had some problems, and when we had no problem as well; we would like to thank the people working there for their most friendly and smiling help.

Last but definitely not least, we received financial help which made possible the conference and the publication of these proceedings; this was provided by the **Dipartimento di Matematica dell'Universitá di Milano** and by the **Universitá di Cagliari**; to these Institutions go our warmest thanks.

Dario Bambusi, Giuseppe Gaeta, Mariano Cadoni
Milano, Roma and Cagliari, July 2001

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3. Special issue of *Acta Applicandae Mathematicae*, to appear

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CONTENTS

[a]For multi-author papers or abstracts, the underlined name corresponds to the author presenting the communication at SPT2001

GEOMETRY AND DYNAMICS OF HYPERELLIPTICALLY SEPARABLE SYSTEMS

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In this paper we focus on the Jacobi–Mumford system and its generalizations.

Many classical integrable systems (like the Euler, Lagrange and Kowalewski tops or the Neumann system) as well as finite dimensional reductions of many integrable PDEs share the property of being algebraically completely integrable systems[4]. This means that they are completely integrable Hamiltonian systems in the usual sense and, moreover, their complexified invariant tori are open subsets of complex Abelian tori on which the complexified flow is linear. To such systems the powerful algebro–geometrical techniques may be applied.

However, the requirement that complexified invariant tori are complex Abelian tori is extremely restrictive and does not include most of Arnold–Liouville integrable systems with algebraic first integrals, the simplest example being the geodesic flow on a triaxial ellipsoid in its natural coordinates[3] as well as certain reductions of integrable PDEs[6,5].

The geodesic flow on the triaxial ellipsoid and finite dimensional reduction of the Harry–Dym hierarchy are typical examples of hyperelliptically separable systems with deficiency[1,2], that is real completely integrable Hamiltonian systems whose generic complexified invariant manifolds are open susbsets of n–dimensional strata of (generalized) hyperelliptic Jacobians (or their coverings). Moreover, we require the existence of coordinates on the (generalized) Jacobian of which n evolve linearly in time and are locally a maximal system of independent coordinates on the stratum. Deficiency is the difference between the dimension of the (generalized) hyperelliptic Jacobian and the dimension of the stratum. In particular, an integrable system is both hyperelliptically separable and algebraically completely integrable if and only if its deficiency is zero.

We now present some geometrical and dynamical properties of hyperelliptically separable systems starting with the classical Neumann system (see for instance Moser[10] and references therein) of a point mass on the N–dimensional unit sphere $S^N \equiv \{\mathbf{q} = (q_1, \ldots, q_{N+1}) \in \mathbf{R}^{N+1} \ : \ q_1^2 + \cdots + q_{N+1}^2 = 1\}$, subject

to the quadratic potential $\mathcal{U}^{(1)} = \sum_{i=1}^{N+1} a_i q_i^2$, where $a_1 < \cdots < a_{N+1}$. The system may be put in Hamiltonian form $H(\mathbf{p},\mathbf{q}) = \frac{1}{2}(p_1^2+\cdots+p_{N+1})+\mathcal{U}^{(1)}(\mathbf{q})$, where $\mathbf{p} = (p_1,\ldots,p_{n+1})$ is the conjugate vector momentum to $\mathbf{q}$ (and we use the canonical Poisson structure). The Neumann system is a completely integrable system in the sense of Arnold–Liouville[7], that is possesses a sufficient number of indepedent first integrals in involution, which we denote $c_0(\mathbf{p},\mathbf{q}) \equiv H(\mathbf{p},\mathbf{q}),\ldots,c_{N-1}(\mathbf{p},\mathbf{q})$, and whose expressions may be obtained from (1) and (4) below. Let

$$\tilde{L}(\lambda) = \Phi(\lambda)\begin{pmatrix} \displaystyle\sum_{i=1}^{N+1}\frac{q_i p_i}{\lambda-a_i}, & \displaystyle\sum_{i=1}^{N+1}\frac{q_i^2}{\lambda-a_i} \\ \displaystyle -1-\sum_{i=1}^{N+1}\frac{p_i^2}{\lambda-a_i}, & \displaystyle -\sum_{i=1}^{N+1}\frac{q_i p_i}{\lambda-a_i} \end{pmatrix}, \tag{1}$$

$$\tilde{A}(\lambda) = \begin{pmatrix} 0, & 1 \\ \displaystyle\lambda+\sum_{i=1}^{N+1}(p_i^2-a_i q_i^2), & 0 \end{pmatrix}, \tag{2}$$

with

$$\Phi(\lambda) = \prod_{i=1}^{N+1}(\lambda-a_i). \tag{3}$$

Then the Neumann system may be put in Lax form

$$\frac{d}{dt}\tilde{L}(\lambda) = [\tilde{L}(\lambda),\tilde{A}(\lambda)].$$

Moreover, upon fixing the constants of motion $c_0,\ldots,c_N$, the characteristic equation

$$\det(\tilde{L}(\lambda)-\mu I) = -\Phi(\lambda)\left(c_0+c_1\lambda+\cdots+c_{N-1}\lambda^{N-1}-\lambda^N\right)-\mu^2 = 0, \tag{4}$$

with $I = diag(1,1)$, defines a genus N hyperelliptic curve Γ (for definitions and properties, see Siegel[13]).

An alternative description is the following one. Let us introduce the spheroconic change of coordinates

$$q_i^2 = \frac{(a_i-\lambda_1)\cdots(a_i-\lambda_n)}{\prod_{j\neq i}(a_i-a_j)}. \tag{5}$$

Then the Hamiltonian takes the Staeckel form[7]

$$H = \frac{1}{2}\sum_{k=1}^{N}\frac{\Phi(\lambda_k)\mu_k^2+\lambda_k^N}{\prod_{j\neq k}(\lambda_k-\lambda_j)},$$

with μ_k conjugate momentum to λ_k, and, upon fixing constants of motion, the equations of motion take the form of Abel–Jacobi differential equations

$$\frac{\lambda_1^k d\lambda_1}{2\sqrt{R(\lambda_1)}} + \cdots + \frac{\lambda_N^k d\lambda_N}{2\sqrt{R(\lambda_N)}} = \delta_{k,N-1} dt, \qquad k = 0, \ldots, N-1, \tag{6}$$

where $R(\lambda) = -\Phi(\lambda)(c_0 + \cdots + c_{N-1}\lambda^{N-1} - \lambda^N)$ and $\mu^2 = R(\lambda)$, is again the affine part of the hyperelliptic curve Γ found in (4). It is easy to check that the N differentials appearing in the left hand side of (6) form a basis of the holomorphic differentials associated to the hyperelliptic curve Γ (for definitions and properties see Siegel[13]).

Moreover, coordinates $(\lambda_1, \mu_1), \ldots, (\lambda_N, \mu_N)$ are points on the curve Γ and the complete image of the N-symmetric product of Γ, $\Gamma^{(N)}$, through the Abel–Jacobi map

$$\phi_k = \sum_{i=1}^{N} \int_{(\lambda_0,\mu_0)}^{(\lambda_i,\mu_i)} \frac{\lambda^{k-1} d\lambda}{2\sqrt{R(\lambda)}}, \qquad k = 1, \ldots, N, \tag{7}$$

with (λ_0, μ_0) fixed basepoint, is the Jacobi variety of Γ, Jac(Γ). Then comparing (6) and (7), we conclude that the closure of the generic complexified invariant manifold is the complex Abelian torus Jac(Γ) and that the flow evolves linearly in time on such complex torus, since $d\phi_1 = \cdots = d\phi_{N-1} = 0$, $d\phi_N = dt$. Following Adler and VanMoerbeke[4], we call the Neumann system algebraically completely integrable or, following Abenda and Fedorov[1], hyperelliptically separable with deficiency zero.

The above construction can be repeated for any hyperelliptically separable system with zero deficiency, as originally shown by Mumford[11] in the odd case (the terms odd and even mean that s is respectively odd or even in $\mu^2 = \prod_{l=1}^{s}(\lambda - e_l)$,). Since the Neumann system is "odd", we just briefly recall the Jacobi–Mumford construction in this case.

Mumford found expressions of coordinates and translationally invariant vector fields on the $3N+1$–dimensional bundle $\mathcal{T}$ over the $2N+1$–dimensional base of odd hyperelliptic curves of genus N, Γ, parametrized by the coefficients of their affine part,

$$\Gamma : \quad \mu^2 = R(\lambda) \equiv \prod_{l=1}^{2N+1} (\lambda - e_l), \tag{8}$$

whose fibers are open subsets of the Jacobi variety Jac(Γ). Indeed, let

$$\begin{aligned} U(\lambda) &= \lambda^N + U_1\lambda^{N-1} + \cdots + U_N, \\ V(\lambda) &= V_1\lambda^{N-1} + \cdots + V_N, \\ W(\lambda) &= \lambda^{N+1} + W_0\lambda^N + \cdots + W_N. \end{aligned} \tag{9}$$

Then the morphism, $\pi : \mathbf{C}^{3N+1} \to \mathbf{C}^{2N+1}$, defined as

$$R(\lambda) = \pi(U(\lambda), V(\lambda), W(\lambda)) \equiv U(\lambda)W(\lambda) + V^2(\lambda),$$

associates the coefficients of a convenient hyperelliptic curve (8) to any choice of coefficients in (9) and the preimage, $\pi^{-1}(R)$, is an open subset of $\mathrm{Jac}(\Gamma)$. Finally, Mumford constructed N commuting vector fields $D_1, \ldots, D_N$ globally defined on $\mathbf{C}^{3N+1}$ and such that they generate the tangent space to $\pi^{-1}(R)$ (that is to $\mathrm{Jac}(\Gamma)$) at each point.

The Jacobi–Mumford system may be put in Lax form

$$\frac{d}{dt}L(\lambda) = [L(\lambda), A(\lambda, \lambda^*)],$$

$$L(\lambda) = \begin{pmatrix} V(\lambda), & U(\lambda) \\ W(\lambda), & -V(\lambda) \end{pmatrix}, \qquad A(\lambda, \lambda^*) = \frac{L(\lambda^*)}{2(\lambda - \lambda^*)} + \frac{1}{2}\begin{pmatrix} 0, & 0 \\ U(\lambda^*), & 0 \end{pmatrix}, \tag{10}$$

where $P^* = (\lambda^*, \mu^*) \in \Gamma$ and the corresponding restriction of the flow to $\mathrm{Jac}(\Gamma)$ is tangent to $P^* \in \Gamma \subset \mathrm{Jac}(\Gamma)$.

In the case of the Neumann system, comparing (1), (2) and (10), we have

$$L(\lambda) = \tilde{L}(\lambda), \qquad A(\lambda, \lambda^*) = \tilde{A}(\lambda),$$

with P^* the infinity point[11].

Many generalizations of such construction have been proposed (see for instance Previato[12] and Beauville[8] for the case of completely algebraically integrable systems associated to r-gonal curves, Novikov and Veselov[15] when fibers are complex tori and Vanhaecke[14] when fibers are symmetric products of algebraic curves).

We now focus on the case of hyperelliptically separable systems with deficiency and, as before, we show the construction explicitly on an example. The Neumann system admits real integrable generalizations on S^N which are hyperelliptically separable with deficiency. Let

$$\mathcal{U}^{(l)}(\mathbf{q}) = -\sum_{j=0}^{\min\{l-1, N-1\}} \mathcal{U}^{(l-j-1)}(\mathbf{q})\tilde{u}_{j+1}(\vec{q}), \quad l \geq 2$$

initialized by $\mathcal{U}^{(0)}(\mathbf{q}) = 1$ and $\mathcal{U}^{(1)}(\mathbf{q}) \equiv \mathcal{U}(\mathbf{q})$, the Neumann potential, with $\tilde{u}_j$'s coefficients of

$$\Phi(\lambda)\sum_{i=1}^{N+1}\frac{q_i^2}{\lambda - a_i} = \lambda^N + \tilde{u}_1(\vec{q})\lambda^{N-1} + \cdots + \tilde{u}_N(\mathbf{q}).$$

Then the generalized Neumann Hamiltonian

$$H^{(l)}(\mathbf{p},\mathbf{q}) = \frac{1}{2}(p_1^2 + \cdots + p_{N+1}) + \mathcal{U}^{(l)}(\mathbf{q}), \quad l \geq 1$$

is completely integrable in Arnold–Liouville sense. The equations of the generalized Neumann system may be put in Lax form for any $l \geq 1$,

$$\frac{d}{dt}\tilde{L}^{(l)}(\lambda) = [\tilde{L}^{(l)}(\lambda), \tilde{A}^{(l)}(\lambda)],$$

where

$$\tilde{L}^{(l)}(\lambda) = \Phi(\lambda)\begin{pmatrix} \displaystyle\sum_{i=1}^{N+1} \frac{q_i p_i}{\lambda - a_i}, & \displaystyle\sum_{i=1}^{N+1} \frac{q_i^2}{\lambda - a_i} \\ -S(\lambda,\mathbf{q}) - \displaystyle\sum_{i=1}^{N+1} \frac{p_i^2}{\lambda - a_i}, & -\displaystyle\sum_{i=1}^{N+1} \frac{q_i p_i}{\lambda - a_i} \end{pmatrix}, \tag{11}$$

with $S(\lambda,\mathbf{q}) = \lambda^{l-1} - \lambda^{l-2}\mathcal{U}^{(1)}(\mathbf{q}) - \cdots - \mathcal{U}^{(l-1)}(\mathbf{q})$, and

$$\tilde{A}^{(l)}(\lambda) = \begin{pmatrix} 0 & 1 \\ \displaystyle\sum_{i=1}^{N+1} p_i^2 - \sum_{0}^{\min\{l,N\}} \mathcal{U}^{l-j}(\mathbf{q})\lambda^{j+1}, & 0 \end{pmatrix}. \tag{12}$$

Using (5), again $H^{(l)}$ takes the Staeckel form and, upon fixing constants of motion, the equations take an Abel–Jacobi like form,

$$\frac{\lambda_1^k d\lambda_1}{2\sqrt{R_l(\lambda_1)}} + \cdots + \frac{\lambda_N^k d\lambda_N}{2\sqrt{R_l(\lambda_N)}} = \delta_{k,N-1}dt, \qquad k = 0,\ldots,N-1, \tag{13}$$

where now $R_l(\lambda) = -\Phi(\lambda)(c_0 + \cdots + c_{N-1}\lambda^{N-1} - \lambda^{N+l-1})$ and

$$\Gamma_l \; : \quad R_l(\lambda) - \mu^2 = \det(\tilde{L}^{(l)}(\lambda) - \mu I) = 0,$$

is the affine part of a genus g hyperelliptic curve Γ_l with

$$g \equiv N + \left[\frac{l-1}{2}\right].$$

If $l \geq 3$, the genus of Γ_l is strictly bigger than the number of holomorphic differentials appearing in the left hand side of (13) and (13) cannot be considered a Abel–Jacobi differential form (since the basis of holomorphic differentials is not complete). We recall that to any genus g hyperelliptic curve there is associated a maximal system of g holomorphic differentials, which may be taken in the form $\dfrac{\lambda^{k-1}d\lambda}{\mu}$, $k = 1,\ldots,g$.

Coordinates $(\lambda_1,\mu_1),\ldots,(\lambda_N,\mu_N)$ are still points on the curve Γ_l. But now, the complete image of the N-symmetric product of Γ_l, $\Gamma_l^{(N)}$, through the Abel–Jacobi map

$$\phi_k = \sum_{i=1}^{N} \int_{(\lambda_0,\mu_0)}^{(\lambda_i,\mu_i)} \frac{\lambda^{k-1}d\lambda}{2\sqrt{R(\lambda)}}, \qquad k=1,\ldots,g=N+\left[\frac{l-1}{2}\right], \tag{14}$$

with (λ_0,μ_0) fixed basepoint, is a N–dimensional analytic subvariety, W_N, of the g–dimensional Jacobi variety of Γ_l, $\mathrm{Jac}(\Gamma_l)$, if $l \geq 3$.

W_N is called *stratum*[9] of $\mathrm{Jac}(\Gamma_l)$. Here we just recall that there exists a natural stratification

$$W_0 \subset W_1 \subset \cdots W_{g-1} \subset W_g \equiv \mathrm{Jac}(\Gamma_l),$$

where W_1 may be identified with the curve Γ_l itself, while W_{g-1} is a copy of the so called theta divisor of $\mathrm{Jac}(\Gamma_l)$.

Comparing (13) and (14), we conclude that the closure of the generic complexified invariant manifold is a stratum of the Jacobi variety.

Finally, we have excessive coordinates $\phi_1,\ldots,\phi_g$ on W_N of which $\phi_1,\ldots,\phi_N$ evolve linearly in time, since $d\phi_1 = \cdots = d\phi_{N-1} = 0$, $d\phi_N = dt$, while the remaining $g-n$, $\phi_{N+1},\ldots,\phi_g$ analytically depend on $\phi_1,\ldots,\phi_n$. Following Abenda and Fedorov[1], we call the generalized Neumann system hyperelliptically separable (and with deficiency if $l \geq 3$).

Let us now generalize the Jacobi–Mumford construction to hyperelliptically separable systems with deficiency. For simplicity, we consider only the case in which the curve is odd and we look for coordinates on the $(2g+N+1)$–dimensional bundle $\mathcal{T}_N$ over the $2g+1$–dimensional base of odd hyperelliptic curves of genus g, Γ, parametrized by the coefficients of their affine part,

$$\Gamma \,:\, \mu^2 = R(\lambda) = \prod_{l=1}^{2g+1}(\lambda - e_l), \tag{15}$$

whose fibers are open subsets of N–dimensional strata, W_N, of $\mathrm{Jac}(\Gamma)$. Let

$$\begin{aligned} U_N(\lambda) &= \lambda^N + U_1^{(N)}\lambda^{N-1} + \cdots + U_N^{(N)}, \\ V_N(\lambda) &= V_1^{(N)}\lambda^{N-1} + \cdots + V_N^{(N)}, \\ W_N(\lambda) &= \lambda^{2g+1-N} + W_0^{(N)}\lambda^{2g-N} + \cdots + W_{2g-N}^{(N)}. \end{aligned} \tag{16}$$

Then, the morphism, $\pi_N : \mathbf{C}^{2g+N+1} \to \mathbf{C}^{2g+1}$,

$$R(\lambda) = \pi_N(U_N(\lambda), V_N(\lambda), W_N(\lambda)) \equiv U_N(\lambda)W_N(\lambda) + V_N^2(\lambda),$$

associates the coefficients of a hyperelliptic curve (15) to any choice of coefficients in (16) and is such that the preimage, $\pi_N^{-1}(R)$, is an open subset of W_N.

Moreover, as first shown by Vanhaecke[14], the Jacobi–Mumford system may be put in Lax form

$$\frac{d}{dt}L^{(N)}(\lambda) = [L^{(N)}(\lambda), A^{(N)}(\lambda, \lambda^*)],$$

setting

$$L^{(N)}(\lambda) = \begin{pmatrix} V^{(N)}(\lambda), & U^{(N)}(\lambda) \\ W^{(N)}(\lambda), & -V^{(N)}(\lambda) \end{pmatrix}, \tag{17}$$

$$A^{(N)}(\lambda, \lambda^*) = \frac{L^{(N)}(\lambda^*)}{2(\lambda - \lambda^*)} + \frac{1}{2}\begin{pmatrix} 0, & 0 \\ U^{(N)}(\lambda^*)\alpha(\lambda), & 0 \end{pmatrix}, \tag{18}$$

where where $P^* = (\lambda^*, \mu^*) \in \Gamma$ and $\alpha(\lambda)$ is a $2(g-N)$–degree polynomial in λ whose coefficients may be recursively computed in function of coefficients in (16) and of λ^*.

Again, the corresponding restriction of the flow to W_N is tangent to $P^* \in \Gamma \subset W_N$ and a maximal system of N independent vector fields may be explicitly constructed which generate the tangent space to $\pi_N^{-1}(R)$ (that is of W_N), at each point.

In the case of the generalized Neumann system, comparing (11), (12), (17) and (18), we have

$$L^{(N)}(\lambda) = \tilde{L}^{(l)}(\lambda), \qquad A^{(N)}(\lambda, \lambda^*) = \tilde{A}^{(l)}(\lambda),$$

with P^* the infinity point.

We end this paper with some remarks. The construction of vector fields of Mumford[11] for hyperelliptically separable systems with zero deficiency is algebro–geometrical and his proof cannot be extended to systems with deficiency (due to obstructions of the Riemann–Roch formula[9]). Vanhaecke[14] directly constructs Hamiltonian systems starting from the N symmetric product of a curve Γ imposing that coefficients of $U^{(N)}(\lambda)$ and $V^{(N)}(\lambda)$ in (16) are Darboux coordinates.

We have completed the Jacobi–Mumford construction for hyperelliptically separable systems with deficiency defining coordinates (16) on all of $\mathcal{T}_N$ and showing that N independent vector fields may be constructed such that they generate the tangent space to $\pi_N^{-1}(R)$ at any point of the stratum. Moreover any integrable system with deficiency may be realized as a convenient Dirac constrained system starting from a convenient integrable system with zero

deficiency. Indeed any point $D \in W_N$ also belongs to Jac(Γ) and the tangent space to W_N at D is a subspace of the tangent space to Jac(Γ) at D. Since the integrable nonlinear flow on W_N may be realized as a convenient restriction of a straight line flow on Jac(Γ) imposing constraints on the phase space variables (see Abenda and Fedorov[3]), then $\mathcal{T}_N$ may be identified as a constrained variety of the fiber space $\mathcal{T}$.

We end pointing out that this unified approach not only has direct consequences in the study of finite dimensional integrable systems, but also it opens new perspectives in the investigation of integrable PDEs whose finite dimensional reductions are integrable systems with deficiency and, possibly, of integrable discrete systems with deficiency too.

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Multiple Hopf bifurcation in problems with $O(2)$ symmetry: Kuramoto-Sivashinky equation

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Abstract

A method to deal with Hopf bifurcation in problems with O(2) symmetry is introduced. Application of the method on Kuramoto-Sivashinsky equation is considered and it is shown that a multiple Hopf bifurcation may occur on a branch with dihedral group of symmetry. This bifurcation is associated with the two dimensional irreducible representation of group D_n.

1 Introduction

Problem with O(2) symmetry often possess a *circle* of nontrivial steady states, each of these states is reflection-symmetric. In addition to reflection symmetry, nontrivial steady states of Kuramoto-Sivashinsky (KS) equation has a discrete rotation symmetry. Therefore we consider Hopf bifurcation which occurs on branches of solutions with D_n symmetry. Due to underlying rotation symmetry the Jacobian of the linearized system, along these branches, is always singular, therefore Hopf bifurcation is not of standard type and usual Hopf theory cannot be applied. The approach of this paper is namely the addition of a phase condition and an extra variable to eliminate the degeneracy due to the group orbit of solutions. We focus on the KS equation [4, 5] and show that bifurcating branches from solutions with Dihedral group of symmetry are either associated with one dimensional irreducible representations of this group giving rise to time periodic solutions with a particular spatio-temporal symmetry, or two dimensional one giving rise to a multiple Hopf bifurcation. The approach enables some of the results of Hyman, Nicolaenko, and Zaleski [6] to be interpreted in a precise way. This problem is considered by Landsberg and Knobloch [1], they eliminate the degeneracy of the system using canonical coordinate transformation [2]. They showed that the bifurcating solutions are rotating waves which are periodic in time and they reverses their direction of propagation in a

periodic manner. This method works perfectly on a small system of ordinary differential equation, however, it has no practical use for KS equation. Krupa [3] considers this type of bifurcation from group orbits in problems with $O(n)$ symmetry. In this case, the vector field were split into two parts, one normal to the group orbit and one tangent to it. The bifurcation analysis are presented on the normal direction and then results given for the whole vector field. Krupa considers KS equation as an example and he shows the same result as given in this paper.

2 Analysis of the Problem

Consider a system of equations of the form

$$\dot{z}(t) = g(z(t), \lambda), \tag{2.1}$$

where $g : X \times \mathbb{R} \longrightarrow X$. We assume that nonlinear function g commutes with the group action $O(2)$ generated by the reflection s and the rotation r_α, where $\alpha \in [0, 2\pi)$. In presence of the reflection s we can decompose the space X as $X = X^s \oplus X^a$, where X^s and X^a are the symmetric and the anti-symmetric spaces with respect to the reflection s, respectively. In problems with O(2) symmetry there are typically many branch of symmetric steady state solutions contained in X^s. Suppose that the trivial solutions with full O(2) symmetry has a bifurcation, associated with the two dimensional irreducible representation of $O(2)$, at $\lambda = 0$ resulting in a branch of nontrivial steady state solutions $z_s = z_s(\lambda)$ contained in $\mathrm{Fix}(Z_2) \times \mathbb{R}$, where $\mathrm{Fix}(Z_2)$ is the fixed point space of vectors which are invariant under $Z_2 = \{I, s\}$. This decomposition implies that the Jacobian matrix of (2.1) has the form $g_z = \mathrm{diag}(g_z^s : g_z^a)$, where g_z^s and g_z^a are associated with symmetric and anti symmetric spaces, respectively [7]. Because of the $O(2)$ symmetry $g_z^a(z_s(\lambda), \lambda)$ has a non-trivial null space, hence $g_z(z_s(\lambda), \lambda)$ has a zero eigenvalue for all λ. Suppose that $g_z^a(z_0, \lambda_0)$ also has eigenvalues $\pm i\omega_0$, where $z_0 = z_s(\lambda_0)$. Note that $\pm i\omega_0$ may occur in symmetric block which is not in our interest or in both blocks and this is the matter of Hopf/Hopf mode interactions considered by Amdjadi and Gomatam [8]. Due to the zero eigenvalue the *standard* Hopf theory cannot be used. The aim is to add a phase condition to the original equation in order to pin down one solution and then use the standard theory.

Wee seek solutions of the form

$$z(t) = r_{ct} x(t), \tag{2.2}$$

where $x(t)$ is time periodic and c is a constant value to allow time periodic solutions drift around the group orbit of solutions. Substituting (2.2) in (2.1) imply the equation of the form

$$\dot{x}(t) = g(x(t), \lambda) - cAx(t) = \tilde{g}(x(t), c, \lambda), \quad A := \frac{dr_\alpha}{d\alpha}|_{\alpha=0}. \tag{2.3}$$

The linear operator $\tilde{g}_x(x(t),0,\lambda)$ is singular along the branch of non-trivial solutions. In order to apply the standard theory the equation (2.3) is extended as

$$\dot{y} = G(y,\lambda) = \begin{bmatrix} \tilde{g}(x(t),c,\lambda) \\ p(x,c,\lambda), \end{bmatrix}, \tag{2.4}$$

where $y = (x,c)$. We want time periodic solutions of (2.4) to correspond to solutions of (2.1) of the form $z(t) = r_{ct}x(t)$ with c constant. Thus we must choose $p(x,c,\lambda)$ such that time periodic solutions of (2.4) give a constant value of c. This is possible if p is independent of time and c. Thus we use $\dot{c} = p(\hat{x},\lambda)$, where $\hat{x}$ is the time average of $x(t)$ over one period T, given by $\hat{x} = \frac{1}{T}\int_0^T x(t)dt$.The periodic boundary conditions imply that $c = constant$. Therefore, the system

$$\dot{y} = G(y,\lambda) = \begin{bmatrix} \tilde{g}(x,c,\lambda) \\ p(\hat{x},\lambda) \end{bmatrix} \tag{2.5}$$

has solution of the form we want. Now, if the phase function $p(\hat{x},c)$ satisfies $p(\widehat{sx},\lambda) = -p(\hat{x},\lambda)$ then $G(y,\lambda)$ is equivariant with respect to S and θ defined by, $S[x,c]^T = [sx,-c]^T$, and $\theta[x,c]^T = [x(t+\frac{T\theta}{2\pi}),c(t+\frac{T\theta}{2\pi})]^T$, with $\theta \in S^1$. A simple choice of p is

$$p(\hat{x},\lambda) = < \ell, \hat{x} > = \frac{1}{T}\int_0^T < \ell, x(t) > dt = 0, \tag{2.6}$$

where $\ell \in X^a$. Other functions could be used based on the work of Jepson and Keller [10].

Lemma 2.1 *The phase condition* (2.6) *will fix the spatial phase of the solution* $(x_0,0,\lambda_0)$ *of* (2.1) *if the following non-degeneracy condition is satisfied:*

$$< p_{\hat{x}}(\hat{x}_0,\lambda_0), A\hat{x}_0 > \neq 0. \tag{2.7}$$

We now consider the eigenvalues of G_y which are related to those of g_x. The following result is given by Dellnitz [9]:

Theorem 2.2 *Suppose that* $(x_0,0,\lambda_0)$ *is a solution of* $G((x,c),\lambda) = 0$. *If the eigenvalues of* $\tilde{g}_x(x_0,0,\lambda_0)$ *are* $\sigma_i, i = 1,...,n$ *with* $\sigma_n = 0$, *then the eigenvalues of* $G_y((x_0,0),\lambda_0)$ *are* σ_i, $i = 1,...,n-1$ *and* $\pm\delta$, *where* $\delta = [- < p_{\hat{x}}(\hat{x}_0,\lambda_0), A\hat{x}_0 >]^{1/2}$.

We note that the non-degeneracy condition (2.7) is precisely $\delta \neq 0$. Clearly, if $\tilde{g}_x(x_0,0,\lambda_0)$ has eigenvalues $\pm i\omega_0$ then so has $G_y((x_0,0),\lambda_0)$.

The Jacobian matrix of equation (2.5) has no zero eigenvalue at a steady state branch of solutions and it has $\pm i\omega_0$ eigenvalues at $(x_0,0,\lambda_0)$. Thus we can apply standard theory to detect a Hopf bifurcation point and to obtain a branch of solutions with symmetry $Z_2 \times S^1$, where Z_2 is generated by S. The additional eigenvalues of $G_y((x_0,0),\lambda_0)$ are $\pm\delta$ where $\delta = [- < \ell, Ax_0 >]^{1/2}$. It is important to choose ℓ so that $< \ell, Ax_0 >$ is negative to ensure that the additional eigenvalues lie on the real axis. Detection of this type of bifurcation

can be achieved using AUTO [11] on the system $G(y,\lambda) = 0$. We note that on the steady state solution, x is independent of time. Thus $\hat{x} = x$ and the phase condition reduces to $< \ell, x >= 0$ which is a simple algebraic equation that is easily implemented.

Once a Hopf bifurcation point has been detected a starting solution on the branch of periodic solutions can be obtained for the variable x using the information contained in the eigenvectors. We note that the eigenvectors of the algebraic equation $G(y,\lambda) = 0$ with the simple phase condition $< \ell, x >= 0$ are not appropriate for constructing the initial solution. This is due to the fact that the linear operator $G_y((x_0,0),\lambda_0)$ is not a constant matrix but involves the time averaging term. We now address this issue. First we linearize the system (2.5) at the steady state solution $y_0 = (x_0, 0)$ to obtain:

$$\dot{\Phi} = G_y(y_0,\lambda_0)\Phi, \tag{2.8}$$

Theorem 2.3

(i) *If g_x^s has eigenvalues $\pm i\omega_0$ then the solution of* (2.8) *is $\Phi(t) = [\Phi_s(t), 0, 0]^T$ where $\dot{\Phi}_s = g_x^s(x_0,\lambda_0)\Phi_s$.*

(ii) *If g_x^a has eigenvalues $\pm i\omega_0$ then the solution of the linearized system* (2.8) *is $\Phi(t) = [0, \Phi_a(t), 0]^T$, where $\Phi_a(t)$ satisfies $\dot{\Phi}_a = g_x^a(x_0,\lambda_0)\Phi_a$ and is constructed using the eigenfunction associated with the eigenvalues $\pm i\omega_0$.*

Note that g_x^s and g_x^a are the block diagonal elements of $\tilde{g}_x(x_0,0,\lambda_0)$ with respect to the spaces X^s and X^a, respectively. The bifurcating solution near the bifurcation point is given by $y(t) = y^s + \alpha\Phi(t) + O(\alpha^2)$. Hence the initial solution is

$$\begin{aligned} x(t) &= x^s + \alpha\Phi_x(t), \\ c &= 0, \\ \lambda &= \lambda_0, \end{aligned}$$

where $\Phi_x(t) = [0, \Phi_a(t)]^T$ and there is no change in λ to first order. To compute the periodic solution the spatial phase condition (2.6) together with a standard temporal phase condition which is built into AUTO can be used. The system will then be solved for x and the scalar variables c and T. The possible further bifurcation may occur in this system with no modifications and so the standard AUTO procedure can be used for detection and swapping branches.

3 Application of the Method to the Kuramoto Sivashinsky Equation

A fairly large number of numerical and theoretical studies have been devoted to the KS equation. The reader is referred to the review paper of Hyman,

Nicolaenko and Zaleski [6]. Of particular interest for our purpose is the existence of symmetry breaking Hopf bifurcations on a steady state branch of solutions which have D_n symmetry, where D_n is the dihedral group generated by the rotation and the reflection. The specific equation we consider has the form

$$v_t + 4v_{xxxx} + \lambda(v_{xx} + vv_x) = 0, \quad v(0,t) = v(2\pi,t) \tag{3.1}$$

where v has zero mean. It is easily verified that this equation is equivariant with respect to the action of $O(2)$ defined by $r_\alpha v(x,t) = v(x+\alpha,t)$, and $sv(x,t) = -v(-x,t)$, where $\alpha \in [0,2\pi)$. Now let X_m be the space of 2π-periodic functions with zero mean whose derivatives up to and including the m^{th} are square integrable. We write equation (3.1) as

$$v_t = g(v,\lambda) = -4v_{xxxx} - \lambda(v_{xx} + vv_x), \quad v \in X := X_4, \ \lambda \in \mathbb{R} \tag{3.2}$$

where $g : X_4 \times \mathbb{R} \longrightarrow X_0$. The steady state of the equation (3.2) has a trivial solution $v = 0$ for $\forall\lambda$ with the full $O(2)$ symmetry. Generically, $\text{Null}(g_v(0,\lambda))$ is irreducible. The non-trivial irreducible representations of the group $O(2)$ acting on the space of 2π-periodic functions are 2-dimensional except for $s = -I$, $r_\alpha = I$. If the action of $O(2)$ on $\phi(x) \in \text{Null}(g_v(0,\lambda))$ gives rise to this representation, the second relation implies that $\phi(x) = c$, where c is an arbitrary constant, therefore $\phi(x) = 0$, since it must have zero mean. Hence generically $\text{Null}(g_v(0,\lambda))$ is two dimensional. It is easy, using Equivariant Branching Lemma [12], to show that bifurcating branches of solutions occur at $\lambda = \lambda_n = 4n^2$, $n \in Z^+$ from the trivial solution with symmetry group D_n. We refer to the n^{th} such branch as primary branch n.

3.1 Dihedral Groups and Multiple Hopf Bifurcation

We now create an extended system of equations by adding a phase condition and we show this system has D_n symmetry. First we substitute a solution of the form $v(x,t) = r_{\alpha(t)}\tilde{v}(x,t)$ into equation (3.2) which, after dropping the tildes, gives

$$v_t + 4v_{xxxx} + \lambda(v_{xx} + vv_x) + c(t)Av = 0, \tag{3.3}$$

where $c(t) = \dot{\alpha}(t)$ and $Av = v_x$. We include the phase condition $\dot{c}(t) =< \ell, \hat{v} >$ to equation (3.3), where ℓ is chosen appropriately to eliminate the group orbit of solutions and $\hat{v}$ is the time average of v over one period, and write it in the form

$$y_t = G(y,\lambda), \tag{3.4}$$

where $y = (v,c) \in Y = X \times \mathbb{R}$. Time periodic boundary conditions imply that $c(t) = constant$. Note that time periodic solution of (3.4) with $c = 0$ correspond to periodic solutions of (3.2) but solutions with $c \neq 0$ correspond to Modulated Travelling Wave (MTW) solutions of (3.2) with drift velocity c (constant). Now, define $S[v,c]^T = [sv,-c]^T$ and $R[v,c]^T = [r_{\frac{2\pi}{n}}v,c]^T$, then $G(y,\lambda)$ is equivariant with respect to D_n generated by R and S if ℓ is chosen such that $r_{\frac{2\pi}{n}}\ell = \ell$ and $s\ell = -\ell$. We refer to solutions of (3.2) which satisfy $sv = v$ as symmetric

solutions. Clearly any solution of $G(y,\lambda)=0$ which satisfies $Sy=y$ must have $c=0$ and thus consists of a symmetric steady state solution of (3.2). If the solution also satisfies $Ry=y$ then the steady state solutions (branch) has D_n symmetry. Note that the full symmetry of the system is $D_n \times S^1$.

We now consider the possibility of time periodic branches of solutions bifurcating from a branch of non-trivial steady states. We assume that along a primary branch of solutions of (3.4), say $y_s = y_s(\lambda)$, there is a point (y_0,λ_0), where $y_0=(v_0,0)$, such that $G_y(y_0,\lambda_0)$ has eigenvalues $\pm i\omega_0$. Often symmetry forces these eigenvalues to be multiple, hence the real eigenspace may have dim >2 so that the standard Hopf theorem cannot be applied. In this case the Equivariant Hopf Theorem [12] must be used. We know that the primary branch $y_s = y_s(\lambda)$ lies in the fixed point space $\mathrm{Fix}(D_n)\times\mathbb{R}$, for some n. Non-trivial irreducible representations of the group D_n are as follows:

$$\begin{array}{llll} i) & R=I,\ S=I, & ii) & R=I,\ S=-I, \\ iii) & R=-I,\ S=I \quad (n \text{ even}), & iv) & R=-I,\ S=-I \quad (n \text{ even}) \end{array}$$

$$v)\quad R=\begin{bmatrix} \cos\frac{2\pi m}{n} & \sin\frac{2\pi m}{n} \\ -\sin\frac{2\pi m}{n} & \cos\frac{2\pi m}{n} \end{bmatrix},\quad S=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where $m=1,\ldots,\frac{1}{2}n-1$ (for n even) and $m=1,\ldots,\frac{1}{2}(n-1)$ (for n odd). The corresponding isotypic components of Y are

$$\begin{array}{ll} i) & Y_0=(\sum_{k=1}^{\infty} a_k \sin nkx, 0)=\mathrm{Fix}(D_n), \\ ii) & Y_1=(\sum_{k=1}^{\infty} a_k \cos nkx, c) \\ iii) & Y_2=(\sum_{k=0}^{\infty} a_k \sin(nk+\frac{n}{2})x, 0) \quad (n \text{ even}), \\ iv) & Y_3=(\sum_{k=0}^{\infty} a_k \cos(nk+\frac{n}{2})x, 0) \quad (n \text{ even}) \\ v) & Y_{4m}=\sum_{k=0}^{\infty}[a_k \sin(nk+m)x + b_k \sin(n(k+1)-m)x], 0) \\ & \quad \oplus(\sum_{k=0}^{\infty}[c_k \cos(nk+m)x + d_k \cos(n(k+1)-m)x], c), \end{array}$$

where $m=1,\ldots,\frac{1}{2}n-1$ (for n even) and $m=1,\ldots,\frac{1}{2}(n-1)$ (for n odd). Thus $Y=Y_0\oplus Y_1\oplus Y_2\oplus Y_3\oplus Y_{4m}$. The Hopf bifurcation can be associated with the one dimensional representations or the two dimensional one. Now we linearize (3.4) as $\dot{\Phi}=G_y(y_0,\lambda_0)\Phi$. In the case of one dimensional representations the solution of the linearized system is $\Phi(x,t)=[\Phi_1(x,t),0]^T$, where $\Phi_1(x,t)=e^{it}\phi_1(x)$ is the solution of the linearization of system (3.2). The eigenfunctions $\phi_1(x)=\phi_{1r}(x)\pm i\phi_{1j}(x)$, with $\phi_{1r}(x)$, $\phi_{1j}(x)\in Y_1$, correspond to the eigenvalues $\pm i$. The linearized system is re-scaled so that $G_y(y_0,\lambda_0)$ has eigenvalues $\pm i$. Thus the corresponding real eigenspace is $E_i=\mathrm{sp}\{\phi_{1r},\phi_{1j}\}\subset Y_1$ and is two dimensional. It is easy to show that $\pi\in S^1$ act as $-I$ on E_i, and since $E_i\subset Y_1$ then $S\phi_1=-\phi_1$, $R\phi_1=\phi_1$ and hence $(S,\pi)=S_1\in D_n\times S^1$ fixes $\phi_1\in E_i$. The action $R_1=(R,0)\in D_n\times S^1$ where $R_1^n=(I,0)=I$ also fixes ϕ_1. Note that $S_1^2=(I,0)$ and $S_1R_1=R_1^{-1}S_1$. Hence, the isotropy group Σ_1 generated by S_1 and R_1 is the symmetry group of the eigenspace E_i and is isomorphic to D_n. Since $\dim(\mathrm{Fix}(\Sigma_1)\cap E_i)=2$, by the Equivariant Hopf Theorem there exists a branch of periodic solutions bifurcating from the steady state branch having Σ_1 as its group of symmetries. Similarly, if $E_i\subset Y_2$ then

the symmetry group Σ_2 of E_i is generated by $S_2 = (S, 0)$ and $R_2 = (R, \pi)$. Note that $R_2^n = (I, 0)$ since n is even and that $S_2 R_2 = R_2^{-1} S_2$ and so again, Σ_2 is isomorphic to D_n. Hence the Equivariant Hopf Theorem implies a branch of solutions with symmetry $\Sigma_2 \subset D_n \times S^1$. Finally if $E_i \subset Y_3$ then the symmetry group Σ_3 of E_i is generated by $S_3 = (S, \pi)$, $R_3 = (R, \pi)$ and is again isomorphic to D_n. In all of these cases, the S_i, $i = 1, 2, 3$ symmetry implies that $c = 0$ so that the bifurcating branch of periodic solutions of (3.4) corresponds to periodic solutions of (3.2). However any further bifurcations which break the reflectional symmetry will give rise to modulated travelling solutions of (3.2).

In the case of two dimensional representation the linearized equation $\dot{\Phi} = G_y(y_0, \lambda_0)\Phi$ has solutions of the form $\Phi_j(x, t) = [e^{it}\phi_j(x), 0]^T$, $j = 1, 2$, where the eigenfunctions $\phi_1 = \phi_{1r} \pm i\phi_{1j}$ and $\phi_2 = \phi_{2r} \pm i\phi_{2j}$ with ϕ_{1r}, ϕ_{1j}, ϕ_{2r}, $\phi_{2j} \in Y_{4m}$ correspond to the eigenvalues $\pm i$. Thus the corresponding real eigenspace is $E_i = \mathrm{sp}\{\phi_{1r}, \phi_{1j}, \phi_{2r}, \phi_{2j}\}$ which is four dimensional. Assume $\phi^T(x) = [\phi_{1r}, \phi_{1j}, \phi_{2r}, \phi_{2j}]$ and define $\gamma \in D_n \times S^1$ by $\gamma\phi = \tilde{T}(\gamma)\phi$, then it is easy to show that the matrix $\tilde{T}(\gamma)$ does not satisfy the homomorphism property therefore does not define a representation. Hence we identify the eigenspace E_i with $\mathbf{C}^2$ by

$$(x_1, y_1, x_2, y_2) \longleftrightarrow x_1\phi_{1r}(x) + x_2\phi_{1j}(x) + y_1\phi_{2r}(x) + y_2\phi_{2j}(x), \tag{3.5}$$

where $z_j = x_j + iy_j$, $j = 1, 2$ and $(z_1, z_2) \in \mathbf{C}^2$. We introduce the new coordinates $(\tilde{z}_1, \tilde{z}_2) = (\bar{z}_1 - i\bar{z}_2, z_1 - iz_2)$ [12], so that in these new coordinates θ acts diagonally on $\mathbf{C}^2$. Golubitsky, Stewart and Schaeffer ([12], ch. XVIII) have shown that there are three isotropy subgroups of $D_n \times S^1$ with n odd, acting on $\mathbf{C}^2$, which give two dimensional fixed point spaces and these are given in Table 1. In case (i), $\tilde{z}_1 = \tilde{z}_2$ giving $y_1 = y_2 = 0$ and the iden-

position	Isotropy subgroup	Fixed point space	dim
(i)	$Z_2(S) = \{(I, 0), (S, 0)\}$	$\{(\tilde{z}_1, \tilde{z}_1)\}$	2
(ii)	$Z_2(S, \pi) = \{(I, 0), (S, \pi)\}$	$\{(\tilde{z}_1, -\tilde{z}_1)\}$	2
(iii)	$\tilde{Z}_n = \{(R, -\frac{2\pi}{n})\}$	$\{(\tilde{z}_1, 0)\}$	2

Table 1: Isotropy subgroups of D_n with 2 dimensional fixed point space for n odd.

tification (3.5) implies that $(x_1, 0, x_2, 0) \longleftrightarrow x_1\phi_{1r} + x_2\phi_{1j}$, where in this case $\mathrm{Fix}(\Sigma_1) \cap E_i = \mathrm{sp}\{\phi_{1r}, \phi_{1j}\}$, with $\Sigma_1 = Z_2(S)$ which reduces E_i to two dimensional space. In case (ii), with $\Sigma_2 = Z_2(S, \pi)$, we have $\tilde{z}_1 = -\tilde{z}_2$ which implies that $x_1 = x_2 = 0$ so identification (3.5) gives $(0, y_1, 0, y_2) \longleftrightarrow y_1\phi_{2r} + y_2\phi_{2j}$. Therefore $\mathrm{Fix}(\Sigma_2) \cap E_i = \mathrm{sp}\{\phi_{2r}, \phi_{2j}\}$. In case (iii), $\tilde{z}_2 = 0$ which implies that $x_1 = -y_2$ and $y_1 = x_2$. Thus the identification (3.5) implies that $(x_1, y_1, y_1, -x_1) \longleftrightarrow x_1(\phi_{1r} - \phi_{2j}) + y_1(\phi_{1j} + \phi_{2r})$. Thus the subgroup $\Sigma_3 = \tilde{Z}_n$, reduces the four dimensional space to a two dimensional one and $\mathrm{Fix}(\Sigma_3) \cap E_i = \mathrm{sp}\{\phi_{1j} + \phi_{2r}, \phi_{1r} - \phi_{2j}\}$. Hence generically there are three branches of periodic solutions bifurcating from the primary branch. The

isotropy subgroups $Z_2(S)$ and $Z_2(S,\pi)$ imply that $c = 0$. Therefore there are two branches of periodic solutions bifurcating from the primary branch at the same point. The isotropy subgroup $\tilde{Z}_n$ does not imply $c = 0$. Hence the branch with this symmetry corresponds to modulated travelling solutions of (3.2). These results agree with Krupa [3] where he has shown that there are three branches bifurcating from the steady state where two consist of periodic orbits and the third one consists of two-tori.

3.2 Numerical Implementation

This section is devoted to implement the above results numerically. We use the spectral Galerkin method, and so we approximate $v(x,t)$ by

$$v_N(x,t) = \sum_{k=1}^{N} (a_k(t)\sin kx + b_k(t)\cos kx).$$

Note that there is no constant term as we assume that v has zero mean. Using the Galerkin method, we obtain the following equations

$$\dot{a}_i + < -g(v_N,\lambda) + cAv_N, \sin ix > = 0, \tag{3.6}$$

$$\dot{b}_i + < -g(v_N,\lambda) + cAv_N, \cos ix > = 0, \quad i = 1,...,N \tag{3.7}$$

$$< \ell, \hat{v}_N > = 0, \tag{3.8}$$

where $<,>$ is the inner product on X_0 defined earlier. On the steady state solutions, v_N is independent of time. Thus $\hat{v}_N = v_N$ and the phase condition reduces to a simple algebraic equation $< \ell, v_N >= 0$. On the branch of periodic solutions, we implement it in AUTO as an integral constraint given by $< \ell, \hat{v}_N >= \frac{1}{T}\int_0^T < \ell, v_N > dt = 0$.

There is a Hopf bifurcation on primary branch three which is found by solving equation (3.6) with $c = 0$ and $b_j = 0$, $j = 1,...,N$. When computing solutions on primary branch three, the lowest order term with a non-zero coefficient is $\sin 3x$ and so the function ℓ is chosen to be $-\cos 3x$ ensuring that $< \ell, v_N >= 0$, with $N = 20$, for the symmetric steady state solutions. For this choice $- < \ell, Av_N >= 3a_3$ which is non-zero and positive on the upper bifurcating branch near the trivial solution. Note that $- < \ell, Av_N >$ must be positive so that the additional eigenvalues of $G_y(y_0,\lambda_0)$ are real.

There is a Hopf bifurcation on primary branch three at $\lambda_0 = 66.751$. At this point, there are four eigenvalues on the imaginary axis and so the bifurcation must be associated with the only two dimensional irreducible representation of D_3. The numerical studies of Hyman, Nicolaenko and Zaleski [6] indicate a bifurcation at $\lambda = 67.5$, but they mentioned that this point is not of Hopf type nor a classical homoclinic loop. Our numerical results give a clearer understanding of the solutions in this region. Having detected the Hopf bifurcation, starting solutions for each of the three branches can be constructed using the eigenvectors in each of the two dimensional fixed point spaces, as described in the previous section. These branches are shown together with the primary branch

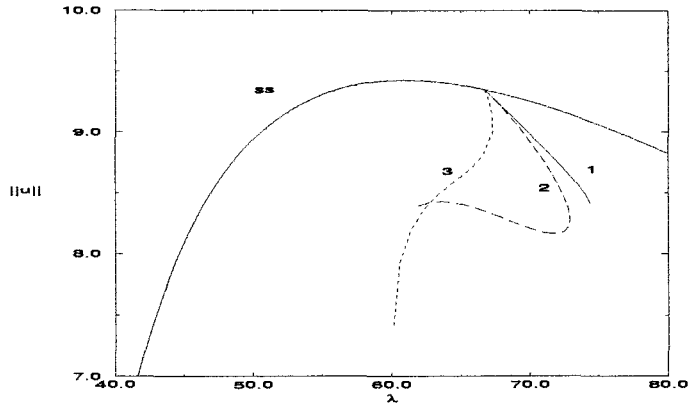

Figure 1: Three branches of solutions bifurcating at $\lambda = 66.751$.

3 in Fig.1. These branches are all locally supercritical and hence one of them must be stable [12]. Numerical results show that there is a turning point at $\lambda = 67.29$ on branch 3. Hyman, Nicolaenko and Zaleski [6] estimated that there is a bifurcation at $\lambda = 67.5$ which is very close to this turning point. Hence we assume that this branch is stable up to the turning point and loses stability at the turning point.

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STERNBERG-CHEN THEOREM FOR EQUIVARIANT HAMILTONIAN VECTOR FIELDS

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Smooth hamiltonian vector fields with linear symmetries and anti-symmetries are considered. We prove that provided symmetry group is compact then a smooth conjugacy in Sternberg-Chen Theorem can be chosen canonical and symmetric.

Introduction

The well-known Sternberg Theorem (see [1]) asserts that if two local smooth vector fields are formally conjugate at a hyperbolic singularity then they are smoothly conjugate. This result reduces local classification and normalization problems to the formal ones.

In the last years there has been a splash of interest to systems with symmetries and anti-symmetries (*reversible systems*) (for references see [2]). This activity faces, in particular, a similar "equivariant" problem: could one provide a conjugacy (smooth, formal) of two symmetric or anti-symmetric vector fields via a transformation keeping the property? The linear and the formal aspects of the problem were considered in [3,4]. In [5] the authors proved a related version of Smooth Conjugacy Sternberg Theorem.

A similar question arised for hamiltonian systems: is it possible to conjugate two hamiltonian vector fields via a *canonical* coordinate change? The affirmative answer was given in [6,7,8].

These two results are in the same streamline and lead to the next very natural setting: given two hamiltonian (anti-)symmetric vector fields can one choose a conjugacy which preserves both symmetric and symplectic structures? The aim of the present paper is to prove a hamiltonian equivariant version of Sternberg-Chen Theorem (Theorem 2.1 below).

We use the so-called *deformation method* or *homotopy trick* going back to J.Mather (see [9,10,8]). This method reduces a nonlinear problem on equiv-

alence of two vector fields (or more generally, of two local mappings near a common singularity) to that on solving a linear functional equation (*cohomology equation*).

1 Definitions

By a *local diffeomorphism* (*local vector field*) we mean in what follows either a germ of diffeomorphisms (vector fields) at a point or a representative of this germ defined in a neighborhood of the point. Sometimes, we omit for briefness the word "local".

Let $\mathfrak{G}$ be a group of local diffeomorphisms $\Phi : (\mathbb{R}^{2d}, 0) \to (\mathbb{R}^{2d}, 0)$ and $\sigma : \mathfrak{G} \to \mathbb{R}$ be a *multiplicative character* of $\mathfrak{G}$, *i.e.*, a homomorphism into the multiplicative group $\mathbb{R}^*$.

Definition 1.1. *A local vector field ξ is said to be $(\mathfrak{G}, \sigma)$-equivariant if*

$$U_*\xi = \sigma(U)\xi \quad (U \in \mathfrak{G}).$$

In this definition, $U_*\xi$ denotes a usual action of the local diffeomorphism U on the vector field ξ:

$$(U_*\xi)(x) = DU(U^{-1}x)\xi(U^{-1}x),$$

where $DU(x)$ is the Jacobi matrix.

A local transformation G keeps the property of ξ to be $(\mathfrak{G}, \sigma)$-equivariant if it commutes with every element $U \in \mathfrak{G}$, *i.e.*, $UG = GU$. Such a local diffeomorphism is said to be $\mathfrak{G}$-*equivariant*.

Let $\mathbb{R}^{2d}$ be provided with the *symplectic structure*, *i.e.*, with a *closed nondegenerate 2-form* ω. The vector field ξ is said to be *symplectic* if $L_\xi\omega = 0$. The *hamiltonian* F, $i_\xi(\omega) = dF$, is a function determined up to a constant. Each function may serve as a hamiltonian of some symplectic vector field.

Let ω be a symplectic form on $\mathbb{R}^{2d}$. According to a theorem due to Darboux, there exist coordinates $(u, v) = (u_1, \ldots, u_d, v_1, \ldots, v_d)$ on $\mathbb{R}^{2d}$ such that

$$\omega(u, v) = du_1 \wedge dv_1 + \ldots + du_d \wedge dv_d.$$

This form is called *standard*. The changes of variables which respect this form, *i.e.*,

$$G^*\omega = \omega$$

are called *canonical*. By $G^*\omega$ we denote a usual action of the local diffeomorphism G on the symplectic form ω.

Definition 1.2. *A symplectic form ω is said to be $(\mathfrak{G},\sigma)$-equivariant if*

$$U^*\omega = \sigma(U)\omega \quad (U \in \mathfrak{G}).$$

In what follows, we assume that $\mathfrak{G}$ is *a compact group of linear operators in* $\mathbb{R}^{2d}$ and σ *is continuous*. Without loss of generality $\mathfrak{G}$ is supposed to be a closed subgroup of the group $O(2d,\mathbb{R})$ of all orthogonal transformaions. Thus, every element $U \in \mathfrak{G}$ keeps the standard inner product in $\mathbb{R}^{2d}$. And besides, the continuity of σ implies

$$\sigma(U) = \pm 1 \quad (U \in \mathfrak{G}).$$

2 Main result

Recall that two local vector fields ξ and η are said to be C^k conjugate if there exists a C^k diffeomorphism G such that

$$G_*\xi = \eta.$$

Take a $(\mathfrak{G},\sigma)$-equivariant symplectic form ω and a local $(\mathfrak{G},\sigma)$-equivariant symplectic C^∞ vector field ξ, $\xi(0) = 0$, $D\xi(0) = \Lambda$. Choose canonical coordinates bringing ω to the standard form. Let the subspaces $E_u = \{v = 0\}$ and $E_v = \{u = 0\}$ be invariant with respect to the phase flow of ξ.

Suppose Λ has no pure imaginary eigenvalues, *i.e.*, ξ has a hyperbolic singularity at the origin. It is known that $\Lambda = IS$, where S is a symplectic linear operator and

$$I = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}.$$

According to Williamson's theorem (see [11]), there exists a canonical linear transformation which brings $\Lambda = IS$ to the form

$$\begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix}$$

where all the eigenvalues of the matrix R have positive real parts.

Denote by δ and Δ the minimum and the maximum of the real parts of the eigenvalues of the matrix R. Given an integer k put

$$\rho(k,\Lambda) = k\frac{\Delta}{\delta} + k + 1.$$

Theorem 2.1. *Let η be a local $(\mathfrak{G},\sigma)$-equivariant symplectic C^K vector field, $\eta(0) = 0$, $K \geq \rho(k,\Lambda)$, and all derivatives of the difference $\xi - \eta$ up to the order K vanish at the origin. Then there is a local $\mathfrak{G}$-equivariant canonical C^k diffeomorphism G which conjugates ξ and η, i.e. $G_*\xi = \eta$.*

In particular, if $K = \infty$ then $k = \infty$ as well.

This theorem is the main result of the paper. Its proof is given in Sections 3 – 4.

3 Deformation method.

As we have mentioned above, the deformation method reduces a conjugacy problem to that of solving a multidimensional linear equation.

Given vector fields ξ and η consider the *deformation* $h(x,\varepsilon) \equiv \varepsilon\eta(x) + (1 - \varepsilon)\xi(x)$ and the vector field

$$\Xi \equiv h(x,\varepsilon)\frac{\partial}{\partial x} + 0 \cdot \frac{\partial}{\partial \varepsilon}$$

on $\mathbb{R}^{n+1}$. Take another vector field

$$\Phi(x,\varepsilon) \equiv \varphi(x,\varepsilon)\frac{\partial}{\partial x} + 1 \cdot \frac{\partial}{\partial \varepsilon}$$

Let

$$Q^t(x,\varepsilon) = (q^t(x,\varepsilon),\varepsilon), \quad P^t(x,\eta) = (p^t(x,\varepsilon),\varepsilon + t)$$

be the pase flows of Ξ and Φ respectively. Put $G(x) = p^1(x,0)$.

Lemma 3.1. *Let the vector fields Φ and Ξ commute, i.e.*

$$[\Phi,\Xi] = 0. \tag{1}$$

Then G conjugates ξ and η, i.e., $G_*\xi = \eta$.

Proof. Note that the phase flows of ξ and η are, respectively, $q_0^t(x) \equiv q^t(x,0)$ and $q_1^t(x) \equiv q^t(x,1)$. Since the phase flows P^t and Q^t commute then

$$\begin{aligned} Q^{t_2}(P^{t_1}(x,\varepsilon)) &= (q^{t_2}(p^{t_1}(x,\varepsilon),\varepsilon + t_1),\varepsilon + t_1) \\ &= (p^{t_1}(q^{t_2}(x,\varepsilon),\varepsilon),\varepsilon + t_1) = P^{t_1}(Q^{t_2}(x,\varepsilon)), \end{aligned}$$

or

$$q^{t_2}(p^{t_1}(x,\varepsilon),\varepsilon + t_1) = p^{t_1}(q^{t_2}(x,\varepsilon),\varepsilon).$$

For $\varepsilon = 0$, $t_1 = 1$, $t_2 = t$ this gives

$$q^t(p^1(x,0),1) = p^1(q^t(x,0),0).$$

For $G(x) = p^1(x,0)$ we get

$$q^t(G(x),1) = G(q^t(x,0)),$$

or

$$q_1^t(G(x)) = G(q_0^t(x)),$$

i.e., G is a conjugacy between ξ and η.

Condition (1) is equivalent to the equation

$$D_1\varphi(x,\varepsilon)\cdot h(x,\varepsilon) = D_1 h(x,\varepsilon)\cdot\varphi(x,\varepsilon) + \eta(x) - \xi(x), \qquad (2)$$

where

$$D_1 \equiv \frac{\partial}{\partial x}.$$

This equation is called the *cohomology equation.*

Now let us turn to the case where the vector fields ξ and η are symplectic. In order to find a canonical conjugacy we make use of the relationship between vector fields and differential forms as suggested in [7].

Let ω be a symplectic form and ξ and η be vector fields which preserve ω. Fix ε and put $h_\varepsilon(x) = h(x,\varepsilon)$, $\varphi_\varepsilon(x) = \varphi(x,\varepsilon)$. It is not hard to verify that

$$i_{[h_\varepsilon,\varphi_\varepsilon]}(\omega) = d(i_{h_\varepsilon}(i_{\varphi_\varepsilon}(\omega))),$$

therefore (2) implies that

$$d(i_{h_\varepsilon}(i_{\varphi_\varepsilon}(\omega))) = i_{\eta-\xi}(\omega).$$

Let X and Y be the hamiltonians of the vector fields ξ and η. Denote by H_ε the hamiltonian of the vector field φ_ε we are seeking. Then

$$d(i_{h_\varepsilon}(dH_\varepsilon)) = dY - dX.$$

Taking into account the equality

$$L_{h_\varepsilon}H_\varepsilon = i_{\xi_\varepsilon}(dH_\varepsilon) + di_{h_\varepsilon}(H_\varepsilon)$$

we get

$$dL_{h_\varepsilon}H_\varepsilon = di_{h_\varepsilon}(dH_\varepsilon),$$

whence

$$d(L_{h_\varepsilon} H_\varepsilon - Y + X) = 0.$$

It suffices to find the hamiltonian H_ε from the equation

$$L_{h_\varepsilon} H_\varepsilon = Y - X$$

or

$$D_1 H(x,\varepsilon) \cdot h(x,\varepsilon) = Y - X \tag{3}$$

which is called the *cohomology equation for hamiltonians.*

Thus, in order to prove that two vector fields which preserve the symplectic form are conjugate via a canonical diffeomorphism we have to solve the cohomology equation (3), then find the corresponding vector field φ_ε and integrate the system $\dot{x} = \varphi_\varepsilon(x)$, $\dot{\varepsilon} = 1$.

4 $\mathfrak{G}$-equivariant conjugacy

First of all, note that solvability of equation (3) under the assumption of Theorem 2.1 was proved in [8] (for $k < \infty$) and [5] (for $k = \infty$). Prove that the solution can be choosed $(\mathfrak{G}, \sigma)$-equivariant.

Let ω be a $(\mathfrak{G}, \sigma)$-equivariant symplectic form. Take a matrix $U \in \mathfrak{G}$,

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are $d \times d$ matrices. Since ω is in the standard form then (see, for example, [12] Proposition 5.5.6) $A^t C$ and $B^t D$ are symmetric and $A^t D - C^t B = \sigma(U) E$. This gives, in particular, that

$$U^{-1} = \sigma(U) \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}.$$

Lemma 4.1. *Let ξ be a symplectic vector field and its hamiltonian F is $(\mathfrak{G}, \sigma)$-equivariant. Then ξ is $\mathfrak{G}$-equivariant.*

Proof. Let F be a $(\mathfrak{G}, \sigma)$-equivariant hamiltonian. Denote

$$\xi(u,v) = \sum_{i=1}^{d} \left(\eta_i(u,v) \frac{\partial}{\partial u_i} + \zeta_i(u,v) \frac{\partial}{\partial v_i} \right).$$

By tansition formulas,

$$\eta_i(u,v) = \frac{\partial F(u,v)}{\partial v_i}, \qquad \zeta_i(u,v) = -\frac{\partial F(u,v)}{\partial u_i} \qquad (i = 1, \ldots, d).$$

Fix a matrix $U \in \mathfrak{G}$. Since $F(u,v) = \sigma(U)F(Au+Bv, Cu+Dv)$ then

$$\eta(u,v) = \sigma(U)(D^t\eta(Au+Bv,Cu+Dv) - B^t\zeta(Au+Bv,Cu+Dv)),$$

$$\zeta(u,v) = \sigma(U)(-C^t\eta(Au+Bv,Cu+Dv) + A^t\zeta(Au+Bv,Cu+Dv)).$$

Thus we conclude that

$$\xi = U^{-1}\,\xi \circ U,$$

i.e., ξ is $\mathfrak{G}$-equivariant.

Lemma 4.2. *If a symplectic vector field ξ with the hamiltonian F is $(\mathfrak{G},\sigma)$-equivariant then F is $\mathfrak{G}$-equivariant.*

Proof. Let ξ be a $(\mathfrak{G},\sigma)$-equivariant symplectic vector field. Fix $U \in \mathfrak{G}$. Since ω is $\mathfrak{G}$-equivariant then

$$\sigma(U)U^{-1} = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}.$$

Therefore,

$$\xi(x) = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix} \xi(Ux).$$

Put $\bar{u} = Au + Bv$, $\bar{v} = Cu + Dv$. Then

$$\begin{aligned} F(u,v) &= \sum_{i=1}^{d} \int_0^1 [\eta^i(ut,vt)v_i - \zeta^i(ut,vt)u_i]dt \\ &= \sum_{i=1}^{d} \int_0^1 [\eta^i(\bar{u}t,\bar{v}t)\bar{v}_i - \zeta^i(\bar{u}t,\bar{v}t)\bar{u}_i]dt = F(Au+Bv,Cu+Dv). \end{aligned}$$

Thus, F is $\mathfrak{G}$-equivariant.

Lemma 4.3. *If H is a C^k solution of equation (3) then for every $U \in \mathfrak{G}$ the function $\tilde{H}$*

$$\tilde{H}(x,\varepsilon) \equiv \sigma(U)H(Ux,\varepsilon),$$

is a solution of equation (3) as well.

Proof. Put $\tilde{H}(x,\varepsilon) \equiv \sigma(U)H(Ux,\varepsilon)$. By virtue of $(\mathfrak{G},\sigma)$-equivariance of ξ and η, the following equalities are true:

$$\begin{aligned} Uh(x,\varepsilon) &= \sigma(U)h(Ux,\varepsilon); \\ (Y-X)(x,\varepsilon) &= (Y-X)(Ux,\varepsilon). \end{aligned}$$

Hence,

$$\begin{aligned} D_1\tilde{H}(x,\varepsilon)h(x,\varepsilon) &= \sigma(U)D_1H(Ux,\varepsilon)Uh(x,\varepsilon) \\ &= \sigma(U)D_1H(Ux,\varepsilon)\sigma(U)h(Ux,\varepsilon) \\ &= D_1H(Ux,\varepsilon)h(Ux,\varepsilon) \\ &= (Y-X)(Ux,\varepsilon) \\ &= (Y-X)(x,\varepsilon). \end{aligned}$$

Therefore, $\tilde{H}$ is a solution of the cohomology equation.

It is well known that since $\mathfrak{G}$ is compact then one can find a Haar measure μ on $\mathfrak{G}$ which is right invariant with respect to $\mathfrak{G}$. We can suppose without loss of generality that $\mu(\mathfrak{G}) = 1$. Let H be a C^k solution of equation (3). Define a C^k function $\mathcal{H} : \mathbb{R}^{2d+1} \to \mathbb{R}$ by

$$\mathcal{H}(x,\varepsilon) = \int_{\mathfrak{G}} \sigma(U)H(Ux,\varepsilon)d\mu(U) \qquad (x \in \mathbb{R}^{2d},\ \varepsilon \in \mathbb{R}). \tag{4}$$

Lemma 4.4. *The function $\mathcal{H}$ is a $(\mathfrak{G},\sigma)$-equivariant solution of the cohomology equation* (3).

Proof. By linearity of the cohomology equation, it follows from Lemma 4.3 that $\mathcal{H}$ is a solution. Since μ is right invariant, for any $V \in \mathfrak{G}$ we have

$$\mathcal{H}(Vx,\varepsilon) = \int_{\mathfrak{G}} \sigma(U)H(UVx,\varepsilon)d\mu(U)$$

$$= \sigma(V)\int_{\mathfrak{G}} \sigma(V)\sigma(U)H(UVx,\varepsilon)d\mu(U) = \sigma(V)\int_{\mathfrak{G}} \sigma(W)H(Wx,\varepsilon)d\mu(W)$$

$$= \sigma(V)\mathcal{H}(x,\varepsilon).$$

Here we put $W = UV$. As a result we get

$$\mathcal{H}(Vx,\varepsilon) = \sigma(V)\mathcal{H}(x,\varepsilon),$$

i.e., $\mathcal{H}$ is $(\mathfrak{G},\sigma)$-equivariant.

Proof of Theorem 2.1. Take a local C^k solution H of equation (3). Then the function $\mathcal{H}$ given by formula (4) is a local $(\mathfrak{G},\sigma)$-equivariant C^k

solution of (3). Let $F^t(x,\varepsilon) \equiv (f^t(x,\varepsilon),\varepsilon + t)$ be the phase flow of the vector field

$$\Psi(x,\varepsilon) = \psi(x,\varepsilon)\frac{\partial}{\partial x} + 1 \cdot \frac{\partial}{\partial \varepsilon},$$

where $\psi(\cdot,\varepsilon)$ is a symplectic vector field generated by the hamiltonian $\mathcal{H}(\cdot,\varepsilon)$. Put $G(x) = f^1(x,0)$. Since $[\Psi,\Xi] = 0$ then, by Lemma 3.1, the local C^k diffeomorphism G conjugates ξ and η. Show that G is $\mathfrak{G}$-equivariant and canonical. In fact, since G is a shift along the trajectories of the symplectic phase flow it presrves the symplectic structure. Note that the vector field $\psi(\cdot,\varepsilon)$ is $\mathfrak{G}$-equivariant. Choose $U \in \mathfrak{G}$ and denote $\tilde{f}^t(x,\varepsilon) \equiv U^{-1}f^t(Ux,\varepsilon)$. Then

$$\left.\frac{d\tilde{f}^t(x,\varepsilon)}{dt}\right|_{t=0} = U^{-1}\left.\frac{df^t(Ux,\varepsilon)}{dt}\right|_{t=0} = U^{-1}\psi(Ux,\varepsilon) = \psi(x,\varepsilon).$$

Hence $\tilde{F}^t(x,\varepsilon) \equiv (\tilde{f}^t(x,\varepsilon),\varepsilon + t)$ is also a phase flow of the vector field Ψ. It follows from Uniqueness Theorem that $\tilde{F}^t = F^t$ and $G(x) = U^{-1}G(Ux)$. We conclude that G is $\mathfrak{G}$-equivariant completing the proof of Theorem 2.1.

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A FUNCTIONAL ANALYSIS APPROACH TO ARNOLD DIFFUSION

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Topological instability of action variables in multidimensional nearly integrable Hamiltonian systems is known as Arnold Diffusion. This phenomenon was first pointed out in 1964 by Arnold himself for a model Hamiltonian in his famous paper [1]. For autonomous Hamiltonian systems with two degrees of freedom KAM theory generically implies topological stability of the action variables (i.e. the time-evolution of the action variables for the perturbed system stay close to their initial values for all times). On the contrary, for systems with more than two degrees of freedom, outside a wide range of initial conditions (the so-called "Kolmogorov set" provided by KAM theory), the action variables may undergo a drift of order one in a very long, but finite time called the "diffusion time". After thirty years from Arnold's seminal work attention to Arnold diffusion has been renewed by Chierchia and Gallavotti [6], using KAM theory and geometrical methods, and by Bessi [5], using variational tools.

In this talk we present a complete and self-contained functional analysis approach, developed in collaboration with P. Bolle [2], [3], [4], apt to deal with Arnold Diffusion. We apply our method to study nearly integrable partially isochronous Hamiltonian systems improving known results. More precisely we consider n-harmonic oscillators weakly coupled with a pendulum through purely spatial perturbations described by the Hamiltonian

$$\mathcal{H}_\mu = \omega \cdot I + \frac{p^2}{2} + (\cos q - 1) + \mu f(\varphi, q),$$

where $(\varphi, q) \in \mathbf{T}^n \times \mathbf{T}^1 := (\mathbf{R}^n/2\pi\mathbf{Z}^n) \times (\mathbf{R}/2\pi\mathbf{Z})$ are the angle variables, $(I, p) \in \mathbf{R}^n \times \mathbf{R}^1$ are the action variables and $\mu \geq 0$ is a small real parameter.

When $\mu = 0$ the energy $\omega_i I_i$ of each oscillator is a constant of the motion. The problem of *Arnold diffusion* in this context is whether, for $\mu \neq 0$, there exist motions whose net effect is to transfer $O(1)$-energy from one oscillator to the others. In order to exclude trivial drifts of the actions due to resonance phenomena, we assume a standard diophantine condition for the frequency vector ω, namely

- $(H1)$ There exists $\gamma > 0, \tau > n$ such that $|\omega \cdot k| \geq \gamma/|k|^\tau$, $\forall k \in \mathbf{Z}^n, k \neq 0$.

The existence of Arnold diffusion is usually proved following the mechanism proposed by Arnold [1]. First one notes that for $\mu = 0$ Hamiltonian $\mathcal{H}_\mu$ admits a continuous family of n-dimensional partially hyperbolic invariant tori $\mathcal{T}_{I_0} = \{(\varphi, I, q, p) \in \mathbf{T}^n \times \mathbf{R}^n \times \mathbf{T}^1 \times \mathbf{R}^1 \mid I = I_0,\ q = p = 0\}$ possessing stable and unstable manifolds, called "whiskers" by Arnold, $W^s(\mathcal{T}_{I_0}) = W^u(\mathcal{T}_{I_0}) = \{(\varphi, I, q, p) \in \mathbf{T}^n \times \mathbf{R}^n \times \mathbf{T}^1 \times \mathbf{R}^1 \mid I = I_0,\ p^2/2 + (\cos q - 1) = 0\}$. By $(H1)$ all the unperturbed tori $\mathcal{T}_{I_0}$, with their stable and unstable manifolds, persist, for μ small enough, being just sligthly deformed. The perturbed stable and unstable manifolds (whiskers) $W^s_\mu(\mathcal{T}^\mu_{I_0})$ and $W^u_\mu(\mathcal{T}^\mu_{I_0})$ may split and intersect transversally giving rise to a chain of tori connected by heteroclinic orbits. By a shadowing argument one can then prove the existence of an orbit such that the action variables I undergo a variation of $O(1)$ in a certain time T_d called the *diffusion time.*

In order to prove the existence of diffusion orbits following the previous mechanism one then encounters two different problems: 1) "Shadowing theorem"; 2) "Splitting of the whiskers".

We now briefly outline the results obtained in [2], [3],[4], throught a Lyapunov-Schmidt type reduction, concerning both problem 1) and 2). Regarding problem 1) we prove by means of a variational technique, once stable and unstable manifolds split, a general shadowing theorem, which improves the known estimates on the diffusion time. Other results like symbolic dynamics can also be obtained by our approach. Concerning problem 2) we introduce a new method for detecting and measuring the splitting of the whiskers (in the difficult case in which "fast frequencies" are present), providing general estimates on the Fourier coefficients of some "splitting function".

As applications we consider the following two cases: (a) the frequencies of the harmonic oscillators form a diophantine vector ω of order 1 ("a priori-unstable case"); (b) the frequencies of the harmonic oscillators form a diophantine vector $\omega_\varepsilon = (1/\sqrt{\varepsilon}, \beta\varepsilon^a)$, $a \geq 0$, $\mu\varepsilon^{-3/2}$ is small, $f(\varphi, q) = (1 - \cos q) f(\varphi)$ and ε is a positive small parameter ("three-time-scales problem").

Case (a) highlights the improvement of our estimates on diffusion times. In this case it is easy to show, using the classical Poincaré-Melnikov function, that the splitting of the whiskers is $O(\mu)$. Then our shadowing method shows that there exist orbits whose action variables undergo a drift of order one, with diffusion time $T_d = O((1/\mu)\log(1/\mu))$. This estimate improves the former results; we underline that it does not depend upon the number of degrees of freedom. Moreover we remark that our shadowing theorem is completely self-contained, in the sense that we do not make use of any KAM-type result for proving, under assumption $(H1)$, the persistence of invariant tori.

The three-time-scales system illustrates an application of our estimate

of the splitting which, together with our general shadowing theorem yields Arnold diffusion in case (*b*). In this case detecting and measuring the splitting of the whiskers is a difficult problem since the frequency vector $\omega = \omega_\varepsilon$ contains the "fast frequency" $\omega_{1,\varepsilon} = 1/\sqrt{\varepsilon}$, ε small. This implies that the Melnikov function along the first direction is exponentially small with respect to ε and then the naive Poincaré-Melnikov expansion provides a valid measure of the splitting only for μ exponentially small with respect to some power of ε. The situation is however very subtle for $\mu = O(\varepsilon^p)$ and $\varepsilon \to 0$, since is general the Poincaré-Melnikov function does not provide the correct prediction of the splitting. However, still by means of a bifurcation technique based on the Lyapunov-Schimdt decomposition, we manage to justify the naive Poincaré-Melnikov approximation for $\mu\varepsilon^{-3/2}$ sufficiently small and hence to describe the non uniform splitting that takes place (naively it is exponentially small along I_1 and polynomially small along the other directions $I_2 \in \mathbf{R}^{n-1}$).

Since larger is the splitting one would expect a faster speed of diffusion one could guess the existence of diffusion orbits that drift along the "fast" directions $I_2 \in \mathbf{R}^{n-1}$, where the splitting is polynomially small, in a polynomially long diffusion time. In [4] we prove that this phenomenon indeed takes place: for the three time scale system (*b*), with $n \geq 3$, exploiting the anisotropy of the splitting, we improve our former shadowing theorem, proving that, along the I_2 directions, Arnold diffusion takes place with fast (polynomial) speed, even though the "splitting determinant" is exponentially small.

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THE SYMPLECTIC EVANS MATRIX AND SOLITARY WAVE INSTABILITY

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Many models for physical phenomena in oceanography, atmospheric dynamics, optical fibre transmission, nerve conduction, acoustical and gas dynamic flows are conservative translation-invariant evolution equations with a Hamiltonian structure. Solitary waves and fronts form an important class of solutions of such equations and the calculus of variations, critical point theory and symplectic structure have played a major role in the analysis of their stability and instability. For example, the characterisation of solitary waves as critical points of the Hamiltonian (energy) constrained to level sets of the momentum (or momentum and other constants of motion) leads to a powerful framework for proving nonlinear Lyapunov stability – when the second variation, evaluated at the constrained critical point, has a finite number of negative eigenvalues (e.g. BENJAMIN[2], BONA[3], HOLM ET AL[14], GRILLAKIS ET AL[12,13], MADDOCKS & SACHS[16] and references therein).

However, for many Hamiltonian evolution equations, particularly coupled systems of PDEs, even though the characterisation of a solitary wave or front solution as a constrained critical point is well-defined, the second variation is strongly indefinite and the relation between critical point type and stability is lost. In this case, an important first step is to study the linear stability and instability, that is, analyse the spectral problem associated with the linearisation about the solitary wave or front solution.

A dynamical systems approach for the analysis of spectral problems associated with the linearisation about a solitary wave or front was first introduced by EVANS[10] in the context of the stability analysis of nerve impulses in mathematical biology. The Evans function framework was substantially generalised by ALEXANDER, GARDNER & JONES[1] to apply to a large class of parabolic PDEs. For more recent results and generalisations see e.g. GARDNER & ZUMBRUN[11], KAPITULA & SANDSTEDE[15] and references therein.

Central to the Evans theory is the *Evans function*, $D(\lambda)$, a complex analytic function of the spectral parameter $\lambda \in \mathbb{C}$. Under suitable hypotheses,

the Evans function has the property that, if $\lambda_0 \in \mathrm{C}$ has positive real part and $D(\lambda_0) = 0$, then λ_0 is an unstable eigenvalue associated with the linearisation about a solitary wave. One way to prove the existence of such unstable eigenvalues is to study the sign of $D(\lambda)$ for λ real when λ is near zero and when λ is large. When the initial-value problem for the PDE is well-posed one can expect that when λ is real and large, $D(\lambda)$ will be of one sign; that is, there would not exist unstable eigenvalues with arbitrarily large growth rate. Assume $D(\lambda) > 0$ for λ large; then a negative sign of the slope of the Evans function for λ near zero can be used to predict the existence of unstable eigenvalues along the real λ axis. The Hamiltonian setting provides a geometrical framework and therefore one can expect to get explicit information about the derivatives of $D(\lambda)$ near $\lambda = 0$ in this setting.

The connection between the Evans function framework and the stability analysis of solitary wave solutions of Hamiltonian evolution equations was first studied by PEGO & WEINSTEIN[17]. For three particular Hamiltonian PDEs they obtained the result that $D(\lambda)$ satisfies $D(0) = D'(0) = 0$, $\operatorname{sign} D''(0) = \operatorname{sign} \frac{\mathrm{d}I}{\mathrm{d}c}$, and $D(\lambda) \to 1$ as $\lambda \to +\infty$ along the real axis. In here, I is the value of the momentum level set and c is the speed of the solitary wave. The system of ODEs associated with the spectral problem had no special structure, requiring explicit calculations in parts of the proof and limiting application to the particular PDEs studied where the solitary wave was known explicitly.

The primary difficulty with an abstract Evans function framework for Hamiltonian evolution equations is that the classical Hamiltonian formulation provides a symplectic structure for time evolution, but much of the analysis of the Evans function is associated with a dynamical system in the x-variable. In BRIDGES & DERKS[7,8] an abstract formulation of the Evans function for Hamiltonian PDEs was proposed based on a multi-symplectic formulation of the PDE, where distinct symplectic operators are assigned for time and space. To be precise, a Hamiltonian system on a multi-symplectic structure will be written in the canonical form

$$\mathbf{M}Z_t + \mathbf{K}Z_x = \nabla S(Z), \quad Z \in \mathrm{R}^{2n}, \quad x \in \mathrm{R}, \quad t \geq 0, \tag{1}$$

where $\mathbf{M}$ and $\mathbf{K}$ are skew-symmetric constant matrices, $S : \mathrm{R}^{2n} \to \mathrm{R}$ is some smooth function and ∇ is a gradient with respect to the standard inner product on R^{2n}.

To demonstrate the multi-symplectification of a Hamiltonian PDE, we consider the (good) Boussinesq equation[4]:

$$u_{tt} = (f(u) - u_{xx})_{xx}, \quad x \in \mathrm{R}, t > 0, \tag{2}$$

where $u(x,t)$ is a real-valued function of x and t and $f(\cdot)$ is some smooth

real-valued function. This system can be written as a classical Hamiltonian system on an infinite-dimensional phase space. For example, let $q = (w, u)^T$ with $w_{xx} = u_t$, then (2) can be reformulated as

$$q_t = \mathbf{J}\frac{\delta H}{\delta q}, \text{ with } \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } H(w,u) = \frac{1}{2}\int (w_x^2 + u_x^2 + f(u))\, dx. \quad (3)$$

However, the phase space is infinite-dimensional and the spatial symplecticity is not explicit in this formulation, but it is implicit in the Hamiltonian function. To formulate this PDE as a multi-symplectic system on a *finite-dimensional phase space*, one can take the Legendre transform of the Hamiltonian function (3). With $v = u_x$ and $z = w_x$ the form (1) is recovered by taking

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{K} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

where $Z = (u, v, w, z)^T$ and $S(Z) = \frac{1}{2}(u^2 - v^2 - z^2) - \frac{1}{3}u^3$.

Many models of physical phenomena, particularly in atmospheric dynamics and optics, are equivariant with respect to a Lie group symmetry. So it is assumed that the system (1) is equivariant with respect to a q-dimensional Abelian subgroup of the Euclidean group acting on R^{2n}, denoted by $\mathcal{G}$, *as well as with respect to spatial translations*. The generators of the group $\mathcal{G}$ are spanned by $\xi_1, \ldots, \xi_q$. According to classical Noether theory for symplectic systems, the symplectic flow of a group generates an invariant function. However, in the multisymplectic setting, there is a flow associated with each symplectic structure which generates a family of functions (cf. BRIDGES[5]). Hence for each generator ξ_i, $i = 1, \ldots, q$, there are functionals P_i and Q_i such that $\mathbf{M}\xi_i(Z) = \nabla P_i(Z)$ and $\mathbf{K}\xi_i(Z) = \nabla Q_i(Z)$.

It is natural to include the symmetries in the definition of the solitary waves or fronts, i.e., a solitary wave/front is a solution of (1) of the form

$$Z(x,t) = G_{\theta(x,t)}\widetilde{Z}(x - ct),$$

where G_θ is the action $\mathcal{G}$ on R^{2n} and $\theta(x,t) = (a_1 t + b_1 x, \ldots, a_q t + b_q x)$. Substitution in the multi-symplectic framework (1) shows that the shape $\widetilde{Z}$ is a homoclinic or heteroclinic orbit of the Hamiltonian ODE

$$(\mathbf{K} - c\mathbf{M})\widetilde{Z}_x = \nabla V(\widetilde{Z}), \quad (4)$$

where $V(Z) = [S - \sum_{i=1}^{q}(a_i P_i + b_i Q_i)](Z)$. The shape of the wave $\widetilde{Z}$ is biasymptotic to an invariant manifold at infinity which is the $\mathcal{G}$-group orbit of a fixed point of (4).

The linearisation of (1) about this solitary wave reduces to a linear ODE of the form

$$U_x = \mathbf{A}(x, \lambda; \mathbf{p})\, U\,, \quad U \in \mathrm{C}^{2n}\,, \quad \lambda \in \mathrm{C}\,, \tag{5}$$

where $\lambda \in \mathrm{C}$ is the spectral parameter, $\mathbf{p}$ represents parameters a_i, b_i and c and $\mathbf{A}(x, \lambda; \mathbf{p}) := (\mathbf{K} - c\mathbf{M})^{-1}[D^2 V(\widetilde{Z}(x; \mathbf{p})) - \lambda \mathbf{M}]$. Central to the Evans function theory are the systems at infinity, defined by

$$\mathbf{A}^{\pm}(\lambda; \mathbf{p}) = \lim_{x \to \pm\infty} \mathbf{A}(x, \lambda; \mathbf{p})\,.$$

Associated with this parameter dependent matrix are the subspaces

$$E^s_{\pm}(\lambda; \mathbf{p}) = \{\, \xi \in \mathrm{C}^{2n} \;:\; \lim_{x \to +\infty} e^{\mathbf{A}^{\pm}(\lambda; \mathbf{p})x} \xi = 0 \,\}\,, \qquad \lambda \in \mathrm{C} \tag{6}$$

$$E^u_{\pm}(\lambda; \mathbf{p}) = \{\, \xi \in \mathrm{C}^{2n} \;:\; \lim_{x \to -\infty} e^{\mathbf{A}^{\pm}(\lambda; \mathbf{p})x} \xi = 0 \,\}\,, \qquad \lambda \in \mathrm{C}, \tag{7}$$

and $E^c_{\pm}(\lambda; \mathbf{p})$, which is defined to be a complement of $E^s_{\pm}(\lambda; \mathbf{p}) \oplus E^u_{\pm}(\lambda; \mathbf{p})$ in C^{2n}. For definiteness, the following properties on the dimension of the systems at infinity are taken: for fixed values of the parameters $\mathbf{p}$,

$$\dim E^s_{\pm}(0; \mathbf{p}) = \dim E^u_{\pm}(0; \mathbf{p}) = 1\,,$$

(hence $\dim E^c_{\pm}(0; \mathbf{p}) = 2n - 2$) and, when $\lambda \neq 0$, there is some $1 \leq p \leq n$ such that

$$\min\{\, \dim E^s_{\pm}(\lambda; \mathbf{p}), \dim E^u_{\pm}(\lambda; \mathbf{p}) \,\} = p\,,$$

for all $\lambda \in \Lambda$, where Λ is subset of C_+, the complex half-plane with positive real part and $0 \in \overline{\Lambda}$. The symplectic structure forces the dimensions of $E^s_{\pm}(0; \mathbf{p})$ and $E^u_{\pm}(0; \mathbf{p})$ to be equal. The property $\dim E^{s,u}_{\pm}(0; \mathbf{p}) = 1$ is not essential and many of the results, such as the construction and definition of the symplectic Evans matrix, are independent of this property.

With these hypotheses on the systems at infinity the Evans function takes the geometric form $D(\lambda; \mathbf{p}) = \det(\mathbf{E}(\lambda; \mathbf{p}))$, where $\mathbf{E}(\lambda; \mathbf{p})$ is the $p \times p$ *symplectic Evans matrix*. Each entry of $\mathbf{E}(\lambda; \mathbf{p})$ is an Ω-symplectic form restricted to a pair of solutions of the linearised stability problem and its adjoint, where Ω is the symplectic form associated with $(\mathbf{K} - c\mathbf{M})$. Hodge duality[6] is the key to transforming the exterior-algebra definition of the Evans function[1] to the Evans matrix. It is follows that $D(0; \mathbf{p}) = \frac{\partial}{\partial \lambda}\big|_{\lambda=0} D(\lambda; \mathbf{p}) = 0$ and $\frac{\partial^2}{\partial \lambda^2}\Big|_{\lambda=0} D(\lambda; \mathbf{p})$ satisfies

$$\operatorname{sign} D''(0) = \Pi \operatorname{sign} \left(\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}c} - \mathcal{B}(c) \right)\,, \tag{8}$$

where $\Pi = \pm 1$ is a geometric sign associated with the shape of the wave, $\mathcal{I}(\widetilde{Z}) = \int_{-\infty}^{\infty}(M\widetilde{Z}_x, \widetilde{Z})\,\mathrm{d}x$ and $\mathcal{B}(c)$ is associated with the properties of the nonconstant manifold of states at infinity (for example, $\mathcal{B}(c) = 0$ for classical solitary waves). All these results combine to give a general instability criterion for a large class of solitary waves and fronts as stated in Theorem 1.

Theorem 1 *For fixed* $\mathbf{p}$, *let* $\lambda_\infty \in \Lambda \cap \mathrm{R}$ *be a positive value of* λ *and let* $d_\infty(\mathbf{p}) = D(\lambda_\infty; \mathbf{p})$. *Define* $Z_0^{\pm}(\mathbf{p}) = \lim_{x \to \pm\infty} \widetilde{Z}(x, \mathbf{p})$ *and*

$$\chi_{00}^{-}(\mathbf{p}) = \lim_{x \to \infty} [\, e^{2\beta(\mathbf{p})x} \Omega(\widetilde{Z}_x(-x; \mathbf{p}), DG_\gamma(Z_0^{-}(\mathbf{p}))^T \widetilde{Z}_x(x; \mathbf{p}))\,]^{-1}.$$

If

$$d_\infty(\mathbf{p})\,\chi_{00}^{-}(\mathbf{p}) \left[\tfrac{\mathrm{d}}{\mathrm{d}c}\mathcal{I}(\widetilde{Z}(x; \mathbf{p})) - \tfrac{1}{2}\,\omega(Z_0^{+}(\mathbf{p}), \partial_c Z_0^{+}(\mathbf{p}))\right] < 0,$$

then the solitary wave or front $G_{\theta(x,t)}\widetilde{Z}(x - ct; \mathbf{p})$ *is linearly spectrally unstable.*

Illustrations of the theory can be found for many examples: a generalised Korteweg-de Vries model from fluid mechanics[7], a Boussinesq model from oceanography[8], a class of nonlinear Schrödinger equations (both coupled and uncoupled) from optics[8], a complex nonlinear Klein-Gordon equation from atmospheric dynamics[8] and a generalised Kawahara equation from plasma waves[9].

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CLASSICAL SYMMETRIES FOR A BOUSSINESQ EQUATION WITH NONLINEAR DISPERSION

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We apply the Lie-group formalism to deduce symmetries of the generalized Boussinesq equation,

$$u_{tt} = au_{xx} + \left(u^{m+1}\right)_{xx} + b\left[u\left(u^{m}\right)_{xx}\right]_{xx},$$

where a and b are arbitrary constants. For $a = 0$ and $m = 1, 2$, this equation describes the vibrations of a purely an harmonic lattice and supports travelling structures with a compact support.

1 Introduction

The Boussinesq equations was extended to include nonlinear dispersion to the effect that the new equations supported compact and semi-compact solitary structures in higher dimensions [1],

$$u_{tt} = au_{xx} + \left(u^{m+1}\right)_{xx} + b\left[u\left(u^{m}\right)_{xx}\right]_{xx}. \tag{1}$$

Eq. (1) describe, for $a = 0$, the vibrations of a purely an harmonic lattice and support travelling structures with a compact support [2].

In this paper we study the generalized Boussinesq equation (1), with a, b, m arbitrary constants and $m \neq 0$, from the point of view of the theory of symmetry reductions for partial differential equations (PDE's). The classical method for finding similarity reductions of (1) is to use the Lie group method of infinitesimal transformations [3]. Though the method is entirely algorithmic, it involves a large amount of tedious algebra and auxiliary calculations which are virtually unmanageable manually. Some symbolic manipulation programs have been developed to simplify the calculations. We use the MACSYMA program symmgrp.max [4]. The fundamental basis of the technique is that, when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists. The machinery of Lie group theory provides the systematic method to search for these special group-invariant solutions. For PDE's with two independent variables, as it is equation (1), a single group reduction transforms the PDE into ordinary differential equations (ODE's), which are generally easier to solve than the original PDE. Most of the required theory and description of the method can be found in the works [3,5].

2 Lie Symmetries

To apply the classical method to Eq. (1) we consider the one-parameter Lie group of infinitesimal transformations in (x,t,u) given by

$$x^* = x + \epsilon\xi(x,t,u) + O(\epsilon^2), \tag{2}$$

$$t^* = t + \epsilon\tau(x,t,u) + O(\epsilon^2),$$

$$u^* = u + \epsilon\eta(x,t,u) + O(\epsilon^2),$$

where ϵ is the group parameter. Then one requires that this transformation leaves invariant the set of solutions of (1). This yields to an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{3}$$

We consider the classical Lie group symmetry analysis of Eq. (1). Invariance of Eq. (1) under a Lie group of point transformations with infinitesimal generator (3) leads to a set of twenty seven determining equations. The solutions of this system depend on the constants of the equation:
Case 1. If a, b and m are arbitrary constants, the only symmetries admitted by (1) are the group of space and time translations, which are defined by the infinitesimal generators

$$V_1 = \lambda\frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}.$$

In the following cases we obtain new symmetries, and these symmetries are defined by the following infinitesimal generators:
Case 2. If m is arbitrary and $a = 0$,

$$V_3^1 = t\frac{\partial}{\partial t} - \frac{2u}{m}\frac{\partial}{\partial u}.$$

Case 3. If a is arbitrary and $m = -1$,

$$V_3^2 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.$$

Case 4. If $a = 0$ and $m = -1$, V_3^2 and

$$V_4 = x\frac{\partial}{\partial x} - 4u\frac{\partial}{\partial u}.$$

3 Optimal system and symmetry reductions

In order to determine solutions of PDE (1) that are not equivalent by the action of the group, we must calculate the one-dimensional optimal system [3]. The generators of the nontrivial one-dimensional optimal system are the set of subalgebras:
Case 2,

$$\{< V_1 >, < \lambda V_1 + V_2 >, < V_3^1 >, < V_1 + V_3^1 >\}.$$

Case 3,

$$\{< V_1 >, < \lambda V_1 + V_2 >, < V_3^2 >\}.$$

Case 4,

$$\{< V_1 >, < V_3^2 >, < V_4 >, < \lambda V_1 + V_2 >, < V_2 + V_4 >, < -V_2 + V_4 >, \\ < V_3^2 + V_4 >, < \lambda V_1 - V_3^2 + V_4 >\}.$$

Having determined the optimal system, the symmetry variables are found by solving the invariant surface condition

$$\Phi \equiv \xi \frac{\partial u}{\partial x} + \tau \frac{\partial u}{\partial t} - \eta = 0. \tag{4}$$

In *case 1*, for $\lambda V_1 + V_2$, we obtain travelling wave reductions

$$z = x - \lambda t, \qquad u = h(z),$$

where $h(z)$ satisfies

$$-bmh^{m+3}h'''' + 2bm\,(2m-1)\,h^{m+2}h'h''' + bm(3m-2)h^{m+2}\,(h'')^2 \\ + \left\{[(m+1)h^m + (a-\lambda^2)]h^3 + mb(6m^2 - 11mb + 5b)h^{m+1}\,(h')^2\right\}h'' \\ + bm(m-2)(m-1)^2h^m\,(h')^4 + (m+1)h^{m+2}\,(h')^2 = 0. \tag{5}$$

Since equation Eq. (1) has additional symmetries and the reductions that correspond to V_1 and V_2 have already been derived, we must only determine the similarity variables and similarity solutions corresponding to the remaining generators:
• For V_3^1 :

$$z = x, \quad u = t^{-\frac{2}{m}}h(z),$$

where $h(z)$ satisfies the ODE

$$-bh^m\left(h'\right)^4m^6-h^m\left(6bh\left(h'\right)^2h''-4b\left(h'\right)^4\right)m^5$$
$$-h^m\left(4bh^2h'h'''+3bh^2\left(h''\right)^2-11bh\left(h'\right)^2h''+5b\left(h'\right)^4+h^2\left(h'\right)^2\right)m^4$$
$$-h^m\left(bh^3h''''-2bh^2h'h'''-2bh^2\left(h''\right)^2+\left(5bh\left(h'\right)^2+h^3\right)h''-2b\left(h'\right)^4\right.$$
$$\left.+h^2\left(h'\right)^2\right)m^3-h^{m+3}h''m^2+2h^4m+4h^4=0.$$

- For $V_1+V_3^1$:

$$z=x-\ln(t),\quad u=t^{-\frac{2}{m}}h(z),$$

where $h(z)$ satisfies the ODE

$$-bh^m\left(h'\right)^4m^6-6bh^{m+1}\left(h'\right)^2h''m^5+4bh^m\left(h'\right)^4m^5-4bh^{m+2}h'h'''m^4$$
$$-3bh^{m+2}\left(h''\right)^2m^4+11bh^{m+1}\left(h'\right)^2h''m^4-5bh^m\left(h'\right)^4m^4-h^{m+2}\left(h'\right)^2m^4$$
$$-bh^{m+3}h''''m^3+2bh^{m+2}h'h'''m^3+2bh^{m+2}\left(h''\right)^2m^3-h^{m+3}h''m^3$$
$$-5bh^{m+1}\left(h'\right)^2h''m^3+2bh^m\left(h'\right)^4m^3-h^{m+2}\left(h'\right)^2m^3-h^{m+3}h''m^2-h^3h''m^2$$
$$-h^3h'm^2-4h^3h'm-2h^4m-4h^4=0.$$

- For V_3^2 :

$$z=\tfrac{x}{t},\quad u=t^{-2}h(z),$$

where $h(z)$ satisfies the ODE

$$bh^3h''''+z^2h^4h''+6zh^4h'-6bh^2h'h'''-5bh^2\left(h''\right)^2+22bh\left(h'\right)^2h''$$
$$-ah^4h''-12b\left(h'\right)^4+6h^5=0.$$

- For V_4 :

$$z=t,\quad u=x^{-4}h(z),$$

where $h(z)$ satisfies the ODE

$$h''-72b=0.$$

- For V_2+V_4 :

$$z=xe^{-t},\quad u=e^{-4t}h(z),$$

where $h(z)$ satisfies the ODE

$$bh^3h''''+z^2h^4h''+9zh^4h'-6bh^2h'h'''-5bh^2\left(h''\right)^2+22bh\left(h'\right)^2h''$$
$$-12b\left(h'\right)^4+16h^5=0.$$

• For $-V_2 + V_4$:

$$z = \frac{e^{-t}}{x}, \quad u = e^{4t}h(z),$$

where $h(z)$ satisfies the ODE

$$bz^8h^3h'''' - 6bz^8h^2h'h''' - 5bz^8h^2(h'')^2 + 22bz^8h(h')^2h'' - 12bz^8(h')^4 + 12bz^8(h')^4 + 12bz^7h^3h''' - 56bz^7h^2h'h'' + 44bz^7h(h')^3 + 36bz^6h^3h'' - 56bz^6h^2(h')^2 + 24bz^5h^3h' + z^2h^4h'' - 7zh^4h' + 16h^5 = 0.$$

• For $V_3^2 + V_4$:

$$z = \frac{x}{t^2}, \quad u = t^{-6}h(z),$$

where $h(z)$ satisfies the ODE

$$bh^3h'''' - 6bh^2h'h''' + 4z^2h^4h'' - 5bh^2(h'')^2 + 22bh(h')^2h'' + 30zh^4h' - 12b(h')^4 + 42h^5 = 0.$$

• For $\lambda V_1 - V_3^2 + V_4$:

$$z = x + \lambda \ln t, \quad u = t^2h(z),$$

where $h(z)$ satisfies the ODE

$$bh^3h'''' - 6bh^2h'h''' + \lambda^2h^4h'' - 5bh^2(h'')^2 + 22bh(h')^2h'' + 3\lambda h^4h' - 12b(h')^4 + 2h^5 = 0.$$

4 Travelling wave solutions

In the following we present the analysis for travelling wave solutions which can be extracted from Eq. (5) for $m = 1$ and $m = 2$.
Integrating Eq. (5) twice with respect to z, we obtain the following cases:
Case 1.1 For $m = 2$,

$$2bh^2h'' + 2bh\,(h')^2 + h^3 + (a - \lambda^2)h + k_1z + k_2 = 0, \tag{6}$$

where k_1 and k_2 are the integrating constants.
• For $k_1 = 0$, the solutions of Eq. (6) are given by

$$\pm 2\sqrt{b} \int \frac{h\,dh}{\sqrt{-h^4 + 2(\lambda^2 - a)h^2 - 4k_2h - 4c_1}} = z + c_2.$$

• For $k_1 = k_2 = 0$, we obtain explicit solutions of Eq. (6) that lead to the following solutions of Eq. (1),

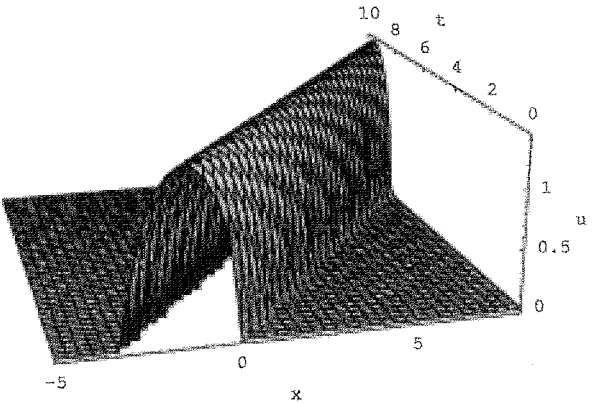

Figure 1. The solution (7) of the Generalized Boussinesq Eq. (1).

Subcase 1.1.1 For $b > 0$,

$$u(x,t) = \sqrt{c_1 \cos\left(\frac{\lambda t - x + c_2}{\sqrt{b}}\right) + \lambda^2 - a}.$$

$$u(x,t) = \sqrt{c_1 \sin\left(\frac{\lambda t - x + c_2}{\sqrt{b}}\right) + \lambda^2 - a}. \tag{7}$$

The plot of the solution (7), for $b = a = c_1 = \lambda = 1$, $c_2 = 0$, and $0 < t - x < \pi$ is given in Figure 1.

Subcase 1.1.2 For $b < 0$,

$$u(x,t) = \sqrt{c_1 \cosh\left(\frac{\lambda t - x + c_2}{\sqrt{-b}}\right) + \lambda^2 - a}.$$

$$u(x,t) = \sqrt{c_1 \sinh\left(\frac{\lambda t - x + c_2}{\sqrt{-b}}\right) + \lambda^2 - a}.$$

Case 1.2 For $m = 1$,

$$bhh'' + h^2 + (a - \lambda^2)h + k_1 z + k_2 = 0, \tag{8}$$

where k_1 and k_2 are the integrating constants.

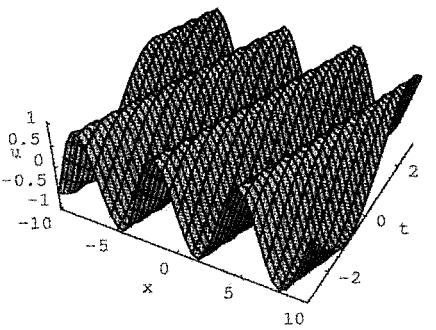

Figure 2. The solution (9) of the Generalized Boussinesq Eq. (1).

• For $k_1 = 0$, the solutions of Eq. (8) are given by

$$\pm\sqrt{b}\int \frac{dh}{\sqrt{-h^2 + 2(\lambda^2 - a)h - 2k_2\ln(h) + 2c_1}} = z + c_2.$$

• For $k_1 = k_2 = 0$, we obtain explicit solutions of Eq. (8) which lead to the following solutions of Eq. (1):
Subcase 1.2.1 If $b < 0$,

$$u(x,t) = c_1 \sinh\left(\frac{x - \lambda t}{\sqrt{-b}}\right) + c_2 \cosh\left(-\frac{x - \lambda t}{\sqrt{-b}}\right) + \lambda^2 - a.$$

Subcase 1.2.2 If $b > 0$,

$$u(x,t) = c_1 \sin\left(\frac{x - \lambda t}{\sqrt{b}}\right) + c_2 \cos\left(\frac{x - \lambda t}{\sqrt{b}}\right) + \lambda^2 - a. \tag{9}$$

Here c_1 and c_2 are constants of integration. The plot of solution (9), for $b = a = c_1 = \lambda^2 = 1$ and $c_2 = 0$, is given in Figure 2.
As, $c_1 \to 0$, $c_2 \to \frac{1}{2}$ and $a \to -\frac{1}{2} + \lambda^2$, these waves turn into compactons

$$u_c(x,t) = \cos^2\left(\tfrac{x - \lambda t}{2\sqrt{b}}\right), \qquad \text{where} \quad |x - t| < \pi. \tag{10}$$

The plot of the solution (10) is given in Figure 3.

5 Conclusions

We have studied the one-dimensional generalized Boussinesq equation (1), by making use of the theory of symmetry reductions in differential equations.

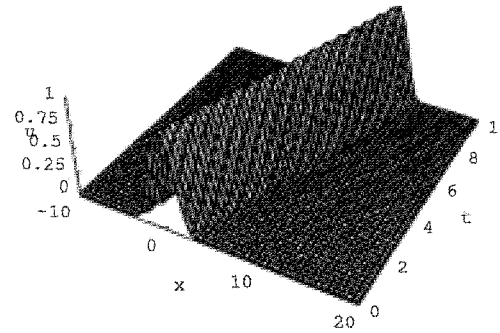

Figure 3. The solution (10) of the Generalized Boussinesq Eq. (1).

We have obtained a complete classification of the Lie symmetries admitted by (1) depending on the values of the constants a, b and m. We have constructed all the invariant solutions with respect to the one-dimensional system of subalgebras, as well as all the ODE's to which (1) is reduced. For $m = 1$ and $m = 2$ we have obtained travelling wave solutions for the Eq. (1).

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PSEUDO-NORMAL FORMS AND THEIR APPLICATIONS

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Introduction and Main Results

Since they were introduced by Poincaré in his thesis, Normal Forms have become a common and useful tool in the local qualitative study of dynamical systems. Consequently, the literature about this subject is very rich, not only because of the people working on it (Poincaré, Dulac, Birkhoff, Stenberg, Chen, Arnold, Moser, Tokarev, Bibikov, Belitskiĭ, Bruno, Walcher, Cicogna, Gaeta, Bambusi and many others) but also for the amount of publications devoted to it (see, for instance [1,2,6,3,7] and references therein).

In order to focus the context of our problem, let us consider a real planar analytic system

$$\dot{z} = F(z) = \Lambda z + \hat{F}(z), \tag{1}$$

with the origin being an equilibrium point and where $\hat{}$ means order greater or equal than 2. It is very well known that a transformation leading system (1) into Poincaré-Dulac normal form (or Birkhoff-Gustavson normal form if we are placed in the Hamiltonian context) does not need to converge in any neighborhood of the origin. In particular, this the case if we assume $\Lambda = DF(0)$ to have purely imaginary eigenvalues, $\pm\alpha i$, $\alpha > 0$. In such situation, the formal normal form becomes of type

$$\begin{cases} \dot{\xi} = \ \xi A(\xi\eta) \\ \dot{\eta} = -\eta A(\xi\eta) \end{cases} \tag{2}$$

where $A(I) = i\alpha + \ldots$ or, in real coordinates, provides we define $r = \xi\eta = \xi\bar{\xi}$, of type

$$\dot{r} = rG(r). \tag{3}$$

In spite of the fact that there always exists an infinitely differentiable change of coordinates normalizing this kind of systems (Tokarev [18], Belitskiĭ [4]), it is also known that such transformation $z = \Phi(\zeta)$ diverges if the origin is a focus (for a detailed discussion about planar normal forms and convergence of the normalizing transformations, see [6]). Quite different is the situation when the

eigenvalues of Λ are real. In this case we have convergence. For instance, for an analytic Hamiltonian system of the plane, this was proved by Moser [14].

In this work, our aim is to approach the problem of the convergence of a normalizing transformation in the case of purely imaginary conjugated eigenvalues. To do it we deal with an *extension* of the classical normal form, which will be convergent even in the case of the origin being a focus. Concretely, this extension is represented by a *remainder term*, depending only on an scalar (analytic) function that contains the obstructions for the integrability of the system. This approach, that comes from ideas of Moser and DeLatte [8], consists in the following: let us first write (Birkhoff) normal form system (2) in the shorter way $\dot{\zeta} = N(\zeta)$, where $N(\zeta) = (\xi A(\xi\eta), -\eta A(\xi\eta))$; to have a (close to identity) change of variables $z = \Phi(\zeta)$ normalizing system (1) implies that we can solve (at least formally) equation $N = \Phi^* F = (D\Phi)^{-1}(F \circ \Phi)$ or, equivalently, $(D\Phi)\, N = F \circ \Phi$. Instead of dealing with it, we look for analytic vector fields N and $\hat{B} = (\xi b(\xi\eta), \eta b(\xi\eta))$, and a change of variables given by Φ in such a way that they satisfy the following equality

$$D\Phi\, N + \hat{B} = F \circ \Phi. \tag{4}$$

We say in this case that Φ leads system (1) into *pseudo-normal form* (ΨNF in shorter). Notice that (4) is not, in general, Poincaré-Dulac normal form, since the term $(D\Phi)^{-1}\,\hat{B}$ in the new system

$$\dot{\zeta} = N(\zeta) + (D\Phi(\zeta))^{-1}\,\hat{B}(\zeta) \tag{5}$$

must not exhibit any particular form *a priori*.

Our first result is that, in this situation, the vector fields N, $\hat{B}$ and such transformation Φ are convergent in a neighborhood of the equilibrium.

However, we prefer to state the main result in a bit more general framework. Namely, the very well known Lyapunov's Theorem [12] ensures that if we have an n-degrees of freedom analytic Hamiltonian system, with the origin being an equilibrium and eigenvalues $\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_n$ satisfying that

- λ_1 is purely imaginary,
- none of the quocients $\frac{\lambda_2}{\lambda_1}, \ldots \frac{\lambda_n}{\lambda_1}$ is an integer,

then there exists a one-parameter family of periodic orbits accumulating to the origin. In other words, one can prove the existence of a transformation leading such system into (Birkhoff) normal form with respect to the variables associated to the imaginary eigenvalue λ_1

In our context, we have the following result, which represents the simplest situation where Lyapunov's Theorem would apply, that is, when the

spectrum of the differential of the field at the equilibrium consists on two pairs of eigenvalues.

Theorem 1 *Let us consider a real system*

$$\begin{cases} \dot{x} = \lambda x + \hat{f}_1(x,y,p,q) \\ \dot{y} = -\lambda y + \hat{g}_1(x,y,p,q) \\ \dot{p} = \alpha q + \hat{f}_2(x,y,p,q) \\ \dot{q} = -\alpha p + \hat{g}_2(x,y,p,q) \end{cases} \tag{6}$$

where $x,y,p,q \in \mathsf{R}$, $\lambda \cdot \alpha \neq 0$ *reals and* $\hat{f}_j$, $\hat{g}_j$, *for* $j = 1,2$, *analytic functions in all their variables which start with terms of order, at least,* 2. *Assume the origin to be a saddle-center equilibrium point of this system. Then, there exists an analytic in a neighborhood of the origin transformation* $X = \Phi(\chi)$, *being* $X = (x,y,p,q)$ *and* $\chi = (\xi,\eta,\mu,\nu)$, *and analytic vector fields* N, $\hat{B}$

$$N = \begin{pmatrix} \xi A_1(\xi\eta, \mu^2+\nu^2) \\ -\eta A_1(\xi\eta, \mu^2+\nu^2) \\ \nu A_2(\xi\eta, \mu^2+\nu^2) \\ -\mu A_2(\xi\eta, \mu^2+\nu^2) \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} \xi b_1(\xi\eta, \mu^2+\nu^2) \\ \eta b_1(\xi\eta, \mu^2+\nu^2) \\ \nu b_2(\xi\eta, \mu^2+\nu^2) \\ \mu b_2(\xi\eta, \mu^2+\nu^2) \end{pmatrix} \tag{7}$$

leading system (6) into ΨNF, *that is, verifying*

$$(D\Phi)\, N + \hat{B} = F \circ \Phi,$$

where $F = (\lambda x + \hat{f}_1, -\lambda y + \hat{g}_1, \alpha q + \hat{f}_2, -\alpha p + \hat{g}_2)$.

In the Hamiltonian case, the remainder term $\hat{B}$ vanishes, so we reobtain, consequently, Lyapunov's result. Moreover, from partial views of system (6) it is not difficult to extend the convergence of the ΨNF procedure to the following cases: a) the origin being a hyperbolic or an elliptic equilibrium of an autonomous system of the plane; b) in a neighborhood of a hyperbolic periodic orbit.

Apart from the class of Hamiltonian systems, it is possible to extract interesting consequences from Theorem 1 if we apply it onto the family of *Reversible* systems. Namely, we say that a system $\dot{X} = F(X)$ is $\mathcal{G}$-*reversible*, $\mathcal{G}$ being an involution ($\mathcal{G}^2 = Id$ and $\mathcal{G} \neq Id$), if it is invariant under $X \longmapsto \mathcal{G}(X)$ and a reversion in the sense of time's arrow ($t \mapsto -t$) (see [11,17,16] and references therein). It turns out that F must satisfy $\mathcal{G}^* F = -F$. Commonly $\mathcal{G}$ is called a *reversing* involution for that system and is, in general, non linear. In this work we assume reversing involutions $\mathcal{G}$ to be analytic. A set which is invariant under the action of $\mathcal{G}$ is called $\mathcal{G}$-*symmetric.*

Thus, in a Hamiltonian \ Reversible context, we have

Theorem 2 *Let us consider an analytic system*

$$\dot{X} = F(X) \tag{8}$$

and assume one of the following three situations holds,

(*i*) $X = (p, q) \in \mathsf{R}^2$ *and the origin is a linear center equilibrium point.*

(*ii*) $X = (x, y, \theta) \in \mathsf{R}^2 \times \mathsf{T}$ *and* $\gamma = \{x = y = 0\}$ *is a (symmetric) hyperbolic periodic orbit.*

(*iii*) $X = (x, y, p, q) \in \mathsf{R}^4$ *and the origin is a saddle-center equilibrium point.*

Then, in a neighborhood of the corresponding equilibrium, the following statements are equivalent

(*i*) *System (8) is Hamiltonian.*

(*ii*) *System (8) is reversible.*

(*iii*) *The analytic vector field* $\hat{B}$ *provided by theorem 1 vanishes. (Notice that is means that the corresponding scalar functions* b_2, b_1 *or both, vanish, respectively).*

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PERIODIC ORBITS OF LANGMUIR'S ATOM

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This is an abstract of a paper, which will be published somwhere else. Our goal is to prove some qualitative results related to a generalization of Langmuir's problem, i.e. that of the classical isosceles 3-body problem (two electrons of equal masses and one nucleus, all assumed to be point particles) whose motion is due to a Coulomb force.

We study an $(n+1)$-body problem ($n \geq 2$ electrons of equal masses and one nucleus). The electrons, which are at the vertices of a regular polygon, move homothetically in a fixed plane such that the polygon increases or decreases but does not change its shape, while the nucleus moves up and down on a line passing through the center of the polygon and perpendicular to its plane. Since we are interested only in dynamical aspects, the point-particle hypothesis is reasonable. Obviously, for $n = 2$ we recover Langmuir's model. Like the isoscels problem, ours also has 2 degrees of freedom.

To make physical sense, a model of the atom needs to satisfy certain qualitative properties. A first condition is that the motion is bounded if not for most solutions then at least for a large class. (If the bound is uniform, the better; this would provide an estimate for the size of the atom.) Then we would also like to see that most solutions in this class are collisionless. (This is not necessary but desirable.) If these properties are satisfied, we can use some recurrence theorem to conclude that most solutions are quasiperiodic. But the most important character of such a model would be that of having many periodic solutions. If periodic orbits exist, semi-classical theory can be used to compute the physical spectrum. In this paper we will show that our model has all the above described qualitative properties.

In Section 1 we present the history of the problem and briefly describe

the main results. In Section 2 we derive the equations of motion and use them in Section 3 to prove that every solution with negative energy $h < 0$ is bounded and that those with $h < h_0$, where $h_0 < 0$ is any fixed energy level, are uniformly bounded. In Section 4 we rewrite the equations of motion in McGehee coordinates, which allow us to analyze the motion near total collision; this analysis is done in Section 5. In Section 6 we prove that a Levi-Civita regularization of the total collapse is possible, therefore the motion can be analytically continued beyond this collision. In Section 7 we show that the system has four symmetries, which form a group isomorphic with that of Klein. This property allows us to show that the system cannot have equally symmetric periodic orbits. In Section 8 we prove that the solutions of the system are both branch and block regularizable and that the homothetic orbits are periodic. In Section 9 we show that on every negative energy level exist at least a finite number of symmetric periodic solutions that are not equally symmetric. This result can be further used for testing the model within the framework of semi-classical theory.

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HETEROCLINIC CYCLES AND WREATH PRODUCT SYMMETRIES

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In previous work[5] we consider the existence and stability of heteroclinic cycles arising from local bifurcation in dynamical systems with wreath product symmetry $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$, where $\mathbf{Z}_2$ acts by ± 1 on $\mathbf{R}$ and $\mathcal{G}$ is a transitive subgroup of the permutation group $\mathbf{S}_N$ (thus $\mathcal{G}$ has degree N). The group Γ acts absolutely irreducibly on $\mathbf{R}^N$. We consider primary (codimension one) bifurcations from an equilibrium to heteroclinic cycles as real eigenvalues pass through zero. We relate the possibility of such cycles to the existence of non-gradient equivariant vector fields of cubic order. Using Hilbert series and the software package MAGMA [1] we show that apart from the cyclic groups $\mathcal{G}$ (already studied by other authors) only five groups $\mathcal{G}$ of degree ≤ 7 are candidates for the existence of heteroclinic cycles. We establish the existence of certain types of heteroclinic cycle in these cases by making use of the concept of a subcycle. We also discuss edge cycles, and a generalisation of heteroclinic cycles which we call a heteroclinic web. We apply our methods to three examples. The work briefly reported here was published in: *Dynamics and Stability of Systems* **15**, 353-385 (2000).

A nonlinear dynamical system possesses a heteroclinic cycle if it has a series of equilibria that are *connected* in the sense that some trajectory links the unstable manifold of any given equilibrium to the stable manifold of the next.

Heteroclinic cycles are especially common in symmetric dynamical systems. Moreover, these may be asymptotically stable and robust (in the sense that they persist under perturbations that respect the symmetry). Asymptotically stable heteroclinic cycles are related to the occurrence of intermittency in applications. The best known example of a robust and asymptotically stable heteroclinic cycle in a symmetric dynamical system is the cycle described by Guckenheimer and Holmes [11]. The system has symmetry $\Gamma = T \oplus \mathbf{Z}_2$, where

T is the tetrahedral group consisting of the orientation-preserving transformations of a tetrahedron. Equivalently, Γ is the semidirect product $\mathbf{Z}_2^3 \dot{+} \mathbf{Z}_3$. In wreath product notation it is $\mathbf{Z}_2 \wr \mathbf{Z}_3$.

Although symmetry can force a heteroclinic cycle to be robust, it may also complicate the description of a cycle, because all symmetrically related equilibria should also be taken into account. Krupa and Melbourne [12] introduce a precise definition of heteroclinic cycles in symmetric systems: in this definition group orbits of equilibria are connected to other group orbits of equilibria. They also prove a sufficient criterium for asymptotic stability of such a cycle.

In [5] we restrict attention to systems with so-called 'wreath product' symmetry, see Golubitsky *et al.* [10], Dionne *et al.* [8] and Dias and Stewart [3,4,6]. To motivate this choice, recall that the symmetry group $\mathbf{Z}_2^3 \dot{+} \mathbf{Z}_3$ of the system studied by Guckenheimer and Holmes is one of the simplest examples of a *wreath product* (a semidirect product of a number of copies of a given group by a group of permutations that permutes those copies). It is $\mathbf{Z}_2 \wr \mathbf{Z}_3$.

The wreath product has very well-behaved algebraic properties, a fact that has persuaded several authors to study wreath products in the hope of finding new examples of robust and stable heteroclinic cycles. One of the main references for the study of heteroclinic cycles in wreath product systems is Field [9], where it is shown that heteroclinic cycles in systems with $\mathbf{Z}_2 \wr \mathbf{Z}_N$ symmetry can be robust asymptotically stable.

All of the symmetry groups mentioned so far are of the type $\mathbf{Z}_2 \wr \mathbf{Z}_N$, and heteroclinic cycles are common in this particular class of systems. In [5] we study local bifurcations from equilibria to robust heteroclinic cycles in a more general (but still special) class of wreath product systems: those for which the symmetry group has the form $\mathbf{Z}_2 \wr \mathcal{G}$, where $\mathcal{G}$ is a transitive subgroup of the permutation group $\mathbf{S}_N$. In this case, we say that the *degree* of $\mathcal{G}$ is N. This choice is motivated by the occurrence of wreath product symmetry in coupled cell systems. There the group $\mathbf{Z}_2$ represents the 'internal' symmetry of a cell and $\mathcal{G}$ the 'global' symmetry of the coupled cell network. We select $\mathbf{Z}_2$ because it corresponds to the simplest nontrivial case, in which each cell has a 1-dimensional dynamic. Also, note that the proof of existence of heteroclinic cycles is difficult when the unstable manifolds of the equilibria involved are high-dimensional. Transitivity of $\mathcal{G}$ may be assumed without loss of generality and simplifies the analysis.

We focus on two main issues: existence of heteroclinic cycles in some appropriate sense, not necessarily that of Krupa and Melbourne [12]; and their asymptotic stability. We show that the choice of the permutation group $\mathcal{G}$ is crucial to the existence and stability of heteroclinic cycles in systems with

symmetry $\mathbf{Z}_2 \wr \mathcal{G}$. We consider here dynamics of equivariant systems under groups $\Gamma = \mathbf{Z}_2^N \dot{+} \mathcal{G}$, with action defined on $\mathbf{R}^N$, and where $\mathcal{G}$ is a transitive subgroup of the permutation group $\mathbf{S}_N$. In wreath product notation $\Gamma = \mathbf{Z}_2 \wr \mathcal{G}$. We show that — at least up to degree 7 — for most non-cyclic transitive groups $\mathcal{G}$, all equivariant vector fields have the same local branching pattern as their cubic truncations, and these truncations are *gradient*. This rules out any recurrent behaviour except equilibria, and in particular it rules out heteroclinic cycles arising by local bifurcation. We use the classification of all transitive permutation groups $\mathcal{G}$ up to degree 15 (up to permutation isomorphism) given in Conway *et al.* [2]

We do, however, find exactly five non-cyclic transitive groups of degree ≤ 7 with non-gradient dynamics, which are thus sensible candidates for the occurrence of heteroclinic cycles.

The gradient property arises in the following manner. Field [9] has introduced the concept of *k-determinacy* of a group, which roughly speaking means that generically all interesting local bifurcation phenomena, for individual vector fields with one bifurcation parameter, are already present in the truncation of the Taylor series of the vector field to order k. Moreover, Field [9] proves that (for the representations we are considering here) the groups $\mathbf{Z}_2 \wr \mathcal{G}$ are 3-determined. Thus, when seeking heteroclinic cycles, we may restrict attention to $\mathbf{Z}_2 \wr \mathcal{G}$-equivariant vector fields truncated at cubic order. We present a method for determining whether a $\mathbf{Z}_2 \wr \mathcal{G}$-equivariant vector field is gradient. The method is based on Hilbert series for rings of invariants (see for example the survey in Dias and Stewart [7]) and leads to a criterium. For most transitive groups $\mathcal{G}$ (up to degree 7) we shall prove that all cubic order truncated vector fields of $\mathbf{Z}_2 \wr \mathcal{G}$ are gradient. We find that apart from cyclic groups, only five groups $\mathcal{G}$ are such that $\mathbf{Z}_2 \wr \mathcal{G}$ possesses non-gradient equivariant vector fields of cubic order.

For the groups that we identify as possessing non-gradient truncated vector fields, we prove that heteroclinic cycles of the type observed by Guckenheimer and Holmes can occur. However, they are always related to a cyclic subgroup of $\mathcal{G}$. We also predict the occurrence of more complicated 'heteroclinic cycles', but in general these cycles are not of the type defined by Krupa and Melbourne [12] in which equilibria are connected only to equilibria in the same group orbit. Instead, each equilibrium in the 'cycle' can be connected to equilibria that need not be in the same group orbit. We call such a configuration a *heteroclinic web*. Heteroclinic webs pose new questions — for example, their existence is a more difficult issue. The stability criterium of Krupa and Melbourne [12] does not directly apply to them, but is easily adapted by Melbourne [13]. We discuss these matters in connection with the example, where

$\mathcal{G}$ is the alternating group $\mathbf{A}_4$ of degree 4 and order 12, but in a permutation representation of degree 6.

We remark that systems of differential equations with wreath product symmetry have been studied previously, especially in the context of symmetric networks of coupled cells ('oscillators') in which each cell has its own symmetry. Our results are therefore of interest for coupled cell systems. The overall symmetry group Γ of such a system depends on the group of *local* (or *internal*) symmetries $\mathcal{L}$ of an individual cell, and on the *global* group $\mathcal{G}$ of permutations of the cells that preserve the network of couplings. One natural type of coupling gives rise to systems with wreath product symmetry: this is the case where the coupling is independent of internal symmetries. In the case under discussion, $\Gamma = \mathcal{L} \wr \mathcal{G}$.

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LINEARIZING RESONANT NORMAL FORMS

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Given a resonant Poincaré-Dulac normal form, we associate to it an auxiliary linear system; solutions to the original system are obtained from solutions to the auxiliary one on a certain invariant submanifold, defined by resonance conditions. If the linearization of the original system satisfies the Poincaré condition, the auxiliary system is finite dimensional.

Introduction

When we deal with dynamical systems with symmetry, it is quite standard to quotient out the symmetry and thus arrive to a reduced dynamical system; when the latter is integrable, e.g. linear or one dimensional, we have actually integrated the original system. In the hamiltonian framework, this is what happens when we use the momentum map to study an integrable system.

It was remarked by Marle[9] and by Kazhdan, Kostant and Sternberg[8] that a different integration procedure is also met: namely, rather than using a symmetry to reduce the system to a lower dimensional one, one lifts the nonlinear system under study to a higher dimensional but linear one. This is what happens, e.g., when we apply the Lax procedure: typically a n-dimensional system is embedded into an n^2-dimensional one (evolution of $(n \times n)$ matrices), as in the integration of the Calogero system.

It was even conjectured[10] that all integrable systems actually originates in higher dimensional linear systems, which we observe in "wrong coordinates", i.e. only through their projection to a lower dimensional nonlinear manifold.

In the present note I show that normal forms corresponding to a resonant Poincaré spectrum can indeed be integrated in this way.

It should be stressed that the integrability of such normal forms is a classical result, already known to Dulac[3]; the point of this note is that the Kazhdan-Kostant-Sternberg-Marle approach applies to this relevant class of equations.

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1 Normal forms

Normal forms[1,2,4,7,11] are central to our understanding of nonlinear dynamics around known solutions, in more ways than we could recall here.

We will consider normal forms of dynamical systems – or, equivalently, vector fields – in $\mathbf{R}^n$ around a regular critical point. We assume the system is already in (or has already been transformed into) normal form, and consider the problem of integrating it. If the linear part of the system is nonresonant, the normal form is linear and the problem is not so interesting; we will thus assume the linear part is *resonant*.

In general the equivalence between a system and its normal form is only formal; however, under conditions on the linear part we can guarantee this equivalence to be analytical. In particular, this is the case if the spectrum of the linear part lies in a *Poincaré domain*; we are thus specially interested in this case and we will analyze it.

1.1 Notation

We will now fix notation and collect several definitions, and properties of normal forms, to be used in the following.

Let us consider a C^∞ function $f : \mathbf{R}^n \to \mathbf{R}^n$ such that $f(0) = 0$, expanded in a series of homogeneous terms $f_k(x)$, where $f_k(ax) = a^{k+1} f_k(x)$ for all real number a. This defines a dynamical system in $V_0 = \mathbf{R}^n$ having a critical point at the origin:

$$\dot{x} \;=\; f(x) \;;\quad x \in \mathbf{R}^n \;,\; f \in C^\infty(\mathbf{R}^n, \mathbf{R}^n) \;,\; f(0) = 0 \tag{1}$$

which we also rewrite, using the expansion in homogeneous terms and singling out the linear part, as

$$\dot{x} \;=\; Ax \;+\; F(x) \;=\; Ax \;+\; \sum_{k=1}^{\infty} f_k(x) \;. \tag{2}$$

We will assume that the linear part of the system, i.e. the matrix A, is in Jordan form.

We consider several vector fields associated to the system and the decomposition given in (2), to be used in the following; we write ∂_i for $\partial/\partial x^i$.

To the function f (i.e. the full dynamical system) we associate the vector field $X_f = f^i(x)\partial_i$; to the linear part $A = (Df)(0)$ we associate the vector field $X_A = (Ax)^i \partial_i$, and to the nonlinear one $X_F = F^i(x)\partial_i$.

We also consider a vector field associated to the adjoint of the matrix A, $X_\ell = (A^+ x)^i \partial_i$; as long as we consider real systems and thus real matrices A, this reduces to $X_\ell = (A^T x)^i \partial_i$.

As well known the matrix A can be decomposed into a semisimple and a nilpotent part, mutually commuting, which we denote as A_s and A_n; as $[A_s, A_n] = 0$, we also have $[A_s, A] = 0 = [A_n, A]$. To the semisimple part A_s of A is associated the vector field $X_0 = (A_s x)^i \partial_i$; we have

$$[X_0, X_A] \;=\; 0 \;=\; [X_0, X_\ell] \;. \tag{3}$$

Finally, we recall that a matrix A is said to be normal if it commutes with its adjoint, $[A, A^+] = 0$; obviously this is equivalent to the condition that $[X_A, X_\ell] = 0$.

1.2 Resonant vectors and monomials

Let $(\lambda_1, ..., \lambda_n)$ be the eigenvalues of A (we denote by σ their ensemble, i.e. the spectrum of A), and let $(\mathbf{e}_1, ..., \mathbf{e}_n)$ be the basis in $V_0 = \mathbf{R}^n$ with respect to which the x^i coordinates are defined. We will use the multiindex notation

$$x^\mu \;:=\; x_1^{\mu_1} ... x_n^{\mu_n} \;. \tag{4}$$

Then the vector $\mathbf{v}_{(\mu)} := x^\mu \mathbf{e}_\alpha$ is resonant with A (or *resonant* for short) if

$$(\mu \cdot \lambda) \;:=\; \sum_{i=1}^{n} \mu_i \lambda_i \;=\; \lambda_\alpha \quad \text{with} \quad \mu_i \geq 0 \;, \quad |\mu| := \sum_{i=1}^{n} \mu_i \geq 2 \;. \tag{5}$$

Note that when we define and determine resonant monomials and vectors, we can as well consider A_s rather than A.

The space of vectors resonant with (the semisimple part of) A is defined as the linear span of such vectors; we will consider a basis $(\mathbf{v}_1, ..., \mathbf{v}_r)$ in this. Thus F is resonant if and only if $F = c_i \mathbf{v}_i$ for some constants c_i.

A monomial x^μ such that (5) is satisfied for some $r \in (1, ..., n)$ is called a *resonant monomial*. We consider a basis of resonant monomials $\{\phi_1(x), ..., \phi_r(x)\}$, and their linear span; this is a linear vector space V_1 (in the space of scalar polynomials on $\mathbf{R}^n$). We choose a basis $\{\mathbf{p}_1, ..., \mathbf{p}_r\}$ in this. Here $\mathbf{p}_i$ corresponds to $\phi_i(x)$, i.e. the scalar polynomial $\sum_{i=1}^r w^i \phi_i(x)$ is represented in V_1 by the vector $\sum_{i=1}^r w^i \mathbf{p}_i$.

1.3 Normal and seminormal forms

We say (see e.g. [4]) that (2) is in Poincaré-Dulac normal form if the vector fields X_ℓ and X_F commute:

$$[\, X_\ell \,,\, X_F \,] \;=\; 0 \;. \tag{6}$$

This implies that all nonlinear terms are resonant with A (i.e. with A_s); however not all resonant terms will satisfy (6) when $A_n \neq 0$, see e.g. example **3** below.

Notice that in general $[X_\ell, X_A] \neq 0$: thus, unless A is normal, we cannot affirm that X_ℓ (or X_A) commutes with X_f.

However it is easy to see from (6) that for systems (2) in Poincaré-Dulac normal form, both X_A and X_F commute with X_0, and therefore

$$[X_0, X_f] = 0 \tag{7}$$

When the system satisfies (7) – although (6) is possibly not satisfied – i.e. if F is resonant with A_s, we say that it is in normal form with respect to A_s or, that it is in *seminormal form*.

As well known, starting from any dynamical system (or vector field) of the form (1), we can arrive to a dynamical system (or vector field) in Poincaré-Dulac normal form by means of a sequence (in general, infinite) of near-identity transformation obtained by means of the Poincaré algorithm; these combine into a near-identity transformation H defined by a series which is in general only formal. The same applies for seminormal forms.

However, one can guarantee the convergence of the series on the basis of properties of the spectrum σ of the linear part A. In particular, it was already known to Poincaré and Dulac that convergence is guaranteed if the convex hull of σ in the complex plane does not include the origin; in this case we say that σ belongs to a Poincaré domain, or that A satisfies the Poincaré condition, or also that σ is a Poincaré spectrum.

It is easy to see – and again was well known to Poincaré and Dulac – that if σ satisfies the Poincaré condition, then only a finite number of resonances is present, i.e. the Poincaré-Dulac normal form is finite (conditions ensuring finiteness of the normal form are discussed in [6]).

2 Map to a linear system

In this section we will give a very elementary procedure to associate to a nonlinear system (2) in resonant normal form an auxiliary linear system in a real vector space $V \simeq V_0 \oplus V_1$ (this can be seen as a trivial bundle $\pi : V \to V_0$ over V_0). In particular, if the normal form is finite then $V = \mathbf{R}^N$ for some finite $N > n$, and this procedure provides a way to explicitely and elementarily integrate the system in normal form.

2.1 General construction

Let V_1, $\phi_i(x)$ and $\mathbf{p}_i$ be as defined in subsection 1.3; we assume for ease of language that r is finite (as mentioned above, this is the case for σ satisfying the Poincaré condition). We consider the real vector space $V = V_0 \oplus V_1 = \mathbf{R}^{n+r}$ with basis vectors $(\mathbf{e}_1, ..., \mathbf{e}_n; \mathbf{p}_1, ..., \mathbf{p}_r)$ and coordinates $\xi = (x^1, ..., x^n; w^1, ..., w^r)$.

We consider a dynamical system $\dot{\xi} = \psi(\xi)$ [a vector field $X_\psi = \psi^j(\xi)(\partial/\partial\xi^j)$] in V defined as follows: first we rewrite (1), (2) substituting $w^1, ..., w^r$ for $\phi_1(x), ..., \phi_r(x)$ (this gives the evolution equation for the x's); then we assign time evolution for the w by $dw^i/dt = (\partial\phi_i(x)/\partial x^j)(dx^j/dt)$. Having written these equations, we will now consider the x and w as indipendent quantities.

It is clear that, by construction, the manifold $\mathcal{M} \subset V$ defined by

$$w^i - \phi_i(x^1, ..., x^n) = 0 \quad \forall i = 1, ..., r \tag{8}$$

is invariant under the flow of the system (the associated vector field) we have defined in this way. On this invariant manifold, the auxiliary system is equivalent to the original one.

The auxiliary evolution equations constructed in this way have several general features in common, mentioned in [5]. The main one is

Proposition. *The evolution equation we obtain for (x, w) is linear.*

Proof. Summation over repeated indices will be understood, and we will denote by $\nu(i)$ the multiindex such that $\nu_j = \delta_{ij}$.

Let us consider a resonant monomial $w = x^\mu$, and say it satisfies $(\lambda \cdot \mu) := \lambda_i \mu_i = \lambda_\alpha$. We denote by $w_{\alpha;i;\sigma}$, $i = 1, ..., q(\alpha)$ the resonant monomials x^σ such that $(\lambda \cdot \sigma) = \lambda_\alpha$ (with the same fixed α). Then we have that (for f in seminormal form) under $\dot{x} = f(x)$ the time evolution of w is given by $X_f(w)$, i.e.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x^i} f^i(x) = \mu_i x^{\mu - \nu(i)} \left[(A_s)^i_j x^j + (A_n)^i_j x^j + c_m w_{\alpha;m;\sigma} \right], \tag{9}$$

where we have used the decomposition (2) and the fact all the nonlinear terms must be resonant.

We assume that A is in Jordan form, so that $A_s = \mathrm{diag}(\lambda_1, ..., \lambda_n)$, and $(A_n)^i_j = \eta_{ij}$ is different from zero (and equal to one) if and only if $j = i+1$ and x^i, x^j belong to the same Jordan block; this implies of course that $\lambda_i = \lambda_j$.

Notice that terms with $\mu_i = 0$ are absent from the sum over i; we can therefore assume $\mu_i \neq 0$. Under this condition, and using the assumption that

A is in Jordan form, we can rewrite

$$\dot{w} \;=\; (\lambda\cdot\mu)\,x^{\mu} \;+\; \mu_i\,\eta_{ij}\,x^{\mu-\nu(i)+\nu(j)} \;+\; c_\sigma\,\mu_i\,x^{\mu-\nu(i)+\sigma}\;. \qquad (10)$$

The first term on the r.h.s. is nothing else that $\lambda_\alpha w$. We want to check that the other terms are also (the sum of) resonant monomials; in order to do this we do not have to worry about the scalar coefficients in fronts of them.

The monomials appearing in the second term of the r.h.s. are of the form $x^{\varphi} = x^{\mu-\nu(j)+\nu(k)}$, and we can assume $\mu_i \neq 0$ and $\eta_{ij} \neq 0$ (or the corresponding monomial would not be present in the sum). For these we have

$$(\lambda\cdot\varphi) \;:=\; \lambda_i\varphi_i \;=\; (\lambda\cdot\mu) - \lambda_i + \lambda_j \;=\; (\lambda\cdot\mu) \;=\; \lambda_\alpha\;; \qquad (11)$$

we have used the fact that $\eta_{ij} \neq 0$ implies $\lambda_i = \lambda_j$, and the resonance relation satisfied by w itself. Thus the second term in the r.h.s. of (8) is the sum of resonant monomials (with the same α as w).

The monomials appearing in the third term are of the form $x^{\varphi} = x^{\mu-\nu(i)+\sigma}$, where $(\lambda\cdot\sigma) = \lambda_i$ and we can assume $\mu_i \neq 0$ (or the corresponding monomial would be absent from the sum). We have now

$$(\lambda\cdot\varphi) \;:=\; \lambda_i\varphi_i \;=\; (\lambda\cdot\mu) - \lambda_i + (\lambda\cdot\sigma) \;=\; \lambda_\alpha\;; \qquad (12)$$

again we have a sum of resonant monomials (with the same α as w).

This concludes the proof that the right hand side of (10) can be written as a linear combination of resonant monomials, i.e. that the evolution equation constructed according to our procedure is linear. Δ

Notice that we have actually proved something more, i.e. that if $w = x^{\mu}$ with $(\lambda\cdot\mu) = \lambda_\alpha$, only resonant monomials $\widetilde{w} = x^{\pi}$ with $(\lambda\cdot\pi) = \lambda_\alpha$ (with the same α as above) will appear in this linear combination. That is, the matrix B will be a block one, where the blocks correspond to resonant monomials identified as described here.

I would also like to stress that if A is not normal, the set of resonant monomials defining vectors in normal form with respect to A^+ would not, in general, be closed under time evolution. This is also immediately seen from the alternative geometrical proof by remarking that if we substitute X_ℓ for X_0, we have (for A not normal) $[X_\ell, X_A] \neq 0$. Thus for non-normal A we cannot limit to consider vectors in normal form, but have to consider the set of all resonant vectors.

3 Integration of normal forms

The strategy to integrate normal forms via the auxiliary linear system $\dot{\xi} = B\xi$ we have defined is rather obvious: this rests on the dynamical invariance of

the manifold defined by (8). That is,

1. For $\xi = (x; w)$, determine the general solution of the linear equation $\dot{\xi} = B\xi$ in $V = \mathbf{R}^{n+r}$, say with solution $\widehat{\xi}(t)$ where $\widehat{\xi}(0) = \xi_0 = (x_0, w_0)$ is the initial datum. This will depend on the $n + r$ arbitrary constants (x_0, w_0).

2. Restrict the general solution to the invariant n-dimensional submanifold $\mathcal{M} \subset \mathbf{R}^{n+r}$ defined by $w^i = \phi_i(x^1, ..., x^n)$. This will depend on the n arbitrary constants x_0.

3. Project the general solution $(x(t), w(t))$ on $\mathcal{M} \subset V$ to the subspace $V_0 = \mathbf{R}^n$ spanned by the x variables, i.e. extract $x(t)$ forgetting about $w(t)$.

The correspondence between the original nonlinear system and the restriction of the auxiliary system to the invariant manifold $\mathcal{M}$ is clear by construction, and projection is globally well defined as $\mathcal{M}$ is identified by the algebraic equations (8). It is therefore clear that this procedure will indeed provide the most general solution to the original nonlinear system in V_0.

This strategy will be particularly simple, and successful, when there is only a finite number of resonances, and in particular when σ belongs to a Poincaré domain.

It should be stressed that if σ belongs to a Poincaré domain, the normal forms could of course also be integrated directly: indeed the corresponding system is nonlinear but, as remarked by Dulac [3], always in triangular form. Namely, we can always write $\dot{x}^i = A^i_j x^j + \Phi^i(x)$ in such a way that $\partial\Phi^i/\partial x^j = 0$ for $j > i$. It is then possible to solve the equations recursively, starting from the linear one for $x^1(t)$ and having at each step a linear equation with a forcing term which is a nonlinear but explicitely known function of t.

The procedure proposed here is equivalent to the one considered by Dulac (and attributed by him to Horn and Lyapounov) from the analytic point of view, but can be more convenient from the computational point of view, as it only requires to solve linear systems (in particular it will be conveniently implemented on computers via algebraic manipulation languages).

It also has the advantage of showing how the nonlinear (normal form) system is obtained by restriction (on the submanifold $\mathcal{M}$) of a linear system via nonlinear constraints, clarifying the connection with topics in modern integrable systems theory [8,10].

Examples of integration of resonant normal forms with Poincaré spectrum (and discussion of the limitation of this approach in other cases) are given elsewhere [5].

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SYMMETRY ANALYSIS AND REDUCTIONS OF THE SCHWARZ-KORTEWEG-DE VRIES EQUATION IN (2+1) DIMENSIONS

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In this paper we make a full analysis of the symmetry reductions of the $(2+1)$-dimensional integrable Schwarz-Korteweg-de Vries equation by using the classical Lie method of infinitesimals. The reduction to systems of partial differential equations in (1+1) are obtained from the optimal system of subalgebras. These systems admit symmetries which lead to further reductions, i.e. to systems of ordinary differential equations. Further, we present a brief analysis for some types of particular solutions.

1 Introduction

The study of higher dimensional integrable system is one of the central themes in integrable systems. In this work we consider the $(2+1)$- dimensional integrable generalization of the Schwarz-Korteweg-de Vries (SKdV) equation

$$W_t+\frac{1}{4}W_{xxz}-\frac{W_xW_{xz}}{2W}-\frac{W_{xx}W_z}{4W}+\frac{W_x^2W_z}{2W^2}-\frac{W_x}{8}\left(\partial_x^{-1}\left(\frac{W_x^2}{W^2}\right)\right)_z=0. \quad (1)$$

Where $\partial^{-1}f=\int f dx$. This equation has been recently derived by Toda and Yu [1] by using the Calogero manner. That is, by modifying one of the operators of the Lax pair for (1+1)-dimensional basic equation to include another spatial variable z. Although this equation arises in a non-local form it can be written by the change $W=\phi_x$, $\phi=exp(\psi)$ and using the further transformations $\psi_x=u$, $\psi_t=v$, as follows

$$\begin{gathered}4u^2v_x-4uu_xv+u^2u_{xxz}-uu_{xx}u_z-3uu_xu_{xz}+3u_x^2u_z-u^4u_z=0,\\ u_t-v_x=0.\end{gathered} \quad (2)$$

To understand the integrability aspects of the system (2), in this work we carry out an invariance analysis and construct the symmetry reductions. Using this analysis we bring out the unexplored invariance properties and similarity reduced systems of $(1+1)$ partial differential equations (PDE's) of the above system (2). First we obtain a point transformation group which leaves the system (2) invariant. In order to find all invariant solutions with respect

to s-dimensional subalgebras, it is sufficient to construct invariant solutions for the optimal system of order s. The set of invariant solutions obtained in this way is called an *optimal system of invariant solutions*. We only consider one-parameter subgroups. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. Although in general this latter problem can still be quite complicated, for one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. The construction of the one-dimensional optimal system appears in [2] using a global matrix for the adjoint transformation. Olver [4], uses a slightly different technique, which we will follow. Using this we construct a table showing the separate adjoint actions of each element in $\mathcal{L}_r$ as it acts on all other elements, this construction is done easily by summing the corresponding Lie series. We then consider a general element in a basis of $\mathcal{L}_r$ and ask whether it can be transformed in a new element of a simpler form by subjecting it, iteratively, to various adjoint transformations. For further details and proofs see [4].

2 Lie symmetries.

In this section we perform Lie symmetry analysis for the $(2+1)$-dimensional system (2). Let us consider a one-parameter Lie group of infinitesimal transformations in (x,t,z,u,v) given by

$$\begin{aligned} x^* &= x + \varepsilon X(x,z,t,u,v) + \mathcal{O}(\varepsilon^2), \\ z^* &= z + \varepsilon Z(x,z,t,u,v) + \mathcal{O}(\varepsilon^2), \\ t^* &= t + \varepsilon T(x,z,t,u,v) + \mathcal{O}(\varepsilon^2), \\ u^* &= u + \varepsilon U(x,z,t,u,v) + \mathcal{O}(\varepsilon^2), \\ v^* &= v + \varepsilon V(x,z,t,u,v) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{3}$$

where ε is the group parameter. Then one requires that this transformation leaves invariant the set of solutions of the system (2). This yields to an overdetermined, linear system of equations for the infinitesimals $X(x,z,t,u,v)$,$Z(x,z,t,u,v)$, $T(x,z,t,u,v)$, $U(x,z,t,u,v)$ and $V(x,z,t,u,v)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = X\frac{\partial}{\partial x} + Z\frac{\partial}{\partial z} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v}. \tag{4}$$

Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface conditions

$$\Phi_1 \equiv X\frac{\partial u}{\partial x}+Z\frac{\partial u}{\partial z}+T\frac{\partial u}{\partial t}-U=0, \qquad \Phi_2 \equiv X\frac{\partial v}{\partial x}+Z\frac{\partial u}{\partial z}+T\frac{\partial v}{\partial t}-V=0. \quad (5)$$

Applying the classical method to the system (2) yields a system of equations which lead to a four-parameter Lie group. Associated with this Lie group we have a Lie algebra which can be represented by the generators, these generators are :

$$\mathbf{v}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{v}_2 = \frac{\partial}{\partial z},$$
$$\mathbf{v}_3 = x\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}, \quad \mathbf{v}_4 = t\frac{\partial}{\partial t} + z\frac{\partial}{\partial z} - v\frac{\partial}{\partial v}.$$

And the infinite-dimensional

$$\mathbf{v}_\alpha = \alpha(t)\frac{\partial}{\partial x} - \alpha'(t)u\frac{\partial}{\partial v}.$$

3 Optimal systems and reductions

In order to construct the one-dimensional optimal system, following Olver, we construct the commutator table (Table 1) and the adjoint table (Table 2) which shows the separate adjoint actions of each element in $\mathbf{v}_i$, $i = 1 \ldots 4$, as it acts on all other elements. This construction is done easily by summing the Lie series.

The corresponding generators of the optimal system of subalgebras are

$$< \mathbf{v}_1 >, < \mu\mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_\alpha >, < \mu\mathbf{v}_2 + \tfrac{1}{2}\mathbf{v}_3 + \mathbf{v}_4 >$$
$$< \mu\mathbf{v}_1 + \mathbf{v}_3 >, < \mu\mathbf{v}_1 + \mathbf{v}_2 >, < \mu\mathbf{v}_3 >, < \mathbf{v}_4 > ,$$

where $\mu \in I\!R^*$ is arbitrary. In the following, we list the corresponding similarity variables and similarity solutions as well as the systems of PDE's obtained when the system (2) is reduced by means of $\{\mathbf{u}_i\}$, $i = 1, \ldots, 6$. These generators are obtained by adding to the generators of the optimal system the infinite dimensional generator $\mathbf{v}_\alpha$.

Reduction 1 By using the generator $\mu\mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$z_1 = xt^{-\mu} - \int t^{-(\mu+1)}\alpha(t)dt, \quad z_2 = zt^{2\mu-1},$$
$$u = t^{-\mu}f(z_1, z_2), \qquad v = t^{-1}(g(z_1, z_2) - f(z_1, z_2)\int t^{-\mu}\alpha'(t)dt),$$

Table 1. Commutator table for the Lie algebra $\{\mathbf{v}_i\}$.

	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	0	0	0	$\mathbf{v}_1$
$\mathbf{v}_2$	0	0	$-2\mathbf{v}_2$	$\mathbf{v}_2$
$\mathbf{v}_3$	0	$2\mathbf{v}_2$	0	0
$\mathbf{v}_4$	$-\mathbf{v}_1$	$-\mathbf{v}_2$	0	0

Table 2. Adjoint table for the Lie algebra $\{\mathbf{v}_i\}$.

Ad	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4 - \varepsilon\mathbf{v}_1$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3 + 2\varepsilon\mathbf{v}_2$	$\mathbf{v}_4 - \varepsilon\mathbf{v}_2$
$\mathbf{v}_3$	$\mathbf{v}_1$	$e^{-2\varepsilon}\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$
$\mathbf{v}_4$	$e^{\varepsilon}\mathbf{v}_1$	$e^{\varepsilon}\mathbf{v}_2$	$\mathbf{v}_3$	$\mathbf{v}_4$

and the systems of PDE's obtained, $\mathbf{S}_1$:

$$4f^2g_{z_1} - 4ff_{z_1}g - ff_{z_1z_1}f_{z_2} + 3f_{z_1}^2f_{z_2} - f^4f_{z_2} + \\ + f^2f_{z_1z_1z_2} - 3ff_{z_1}f_{z_1z_2} = 0,$$
$$(2\mu - 1)z_2f_{z_2} - \mu z_1f_{z_1} - g_{z_1} - \mu f = 0.$$

Reduction 2 By using the generator $\mu\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 + \mathbf{v}_4 + \mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$z_1 = xt^{-1/2} - \int t^{-3/2}\alpha(t)dt,\ z_2 = t^{1/2}e^{-z},$$
$$u = t^{-1/2}f(z_1, z_2), \qquad v = t^{-1}(g(z_1, z_2) - \int t^{-1/2}\alpha'(t)dtf(z_1, z_2)),$$

and the systems of PDE's obtained, $\mathbf{S}_2$:

$$z_2(ff_{z_1z_1}f_{z_2} - 3(f_{z_1})^2f_{z_2} + f^4f_{z_2} - f^2f_{z_1z_1z_2} + \\ + 3ff_{z_1}f_{z_1z_2}) + 4f^2g_{z_1} - 4ff_{z_1}g = 0,$$
$$z_2f_{z_2} - z_1f_{z_1} - 2g_{z_1} - f = 0.$$

Reduction 3 By using the generator $\mu\mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_\alpha$, we obtain the similarity variables and similarity solutions

$$z_1 = xe^{-t/\mu} - \tfrac{1}{\mu}\int e^{t\{\mu}\alpha(t)dt,\ z_2 = ze^{2t/\mu},$$
$$u = e^{-t/\mu}f(z_1, z_2), \qquad v = g(z_1, z_2) - \tfrac{1}{\mu}\int t^{-(\mu+1)}\alpha(t)dtf(z_1, z_2),$$

and the systems of PDE's obtained, $\mathbf{S}_3$:

$$\begin{aligned}&4f^2g_{z_1}-4ff_{z_1}g-ff_{z_1z_1}f_{z_2}+3(f_{z_1})^2f_{z_2}-f^4f_{z_2}+\\&\quad+f^2f_{z_1z_1z_2}-3ff_{z_1}f_{z_1z_2}=0,\\&2z_2f_{z_2}-z_1f_{z_1}-\mu g_{z_1}-f=0.\end{aligned}$$

Reduction 4 By using the generator $\mu\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_\alpha$ we obtain the similarity variables and similarity solutions

$$\begin{aligned}&z_1=x-\tfrac{1}{\mu}\textstyle\int\alpha(t)dt, && z_2=\mu z-t,\\&u=f(z_1,z_2), && v=g(z_1,z_2)-\tfrac{1}{\mu}\alpha f(z_1,z_2),\end{aligned}$$

and the systems of PDE's obtained, $\mathbf{S}_4$:

$$\begin{aligned}&\mu(-ff_{z_1z_1}f_{z_2}+3f_{z_1}^2f_{z_2}-f^4f_{z_2}+\\&\quad+f^2f_{z_1z_1z_2}-3ff_{z_1}f_{z_1z_2})+4f^2g_{z_1}-4ff_{z_1}g=0,\\&g_{z_1}+f_{z_2}=0.\end{aligned}$$

Reduction 5 By using the generator $\mathbf{v}_3+\mathbf{v}_\alpha$ we obtain the similarity variables and similarity solutions

$$\begin{aligned}&z_1=t, && z_2=(x+\alpha(t))^2z,\\&u=z^{1/2}f(z_1,z_2), && v=z^{1/2}\alpha'(t)f(z_1,z_2)+g(z_1,z_2),\end{aligned}$$

and the systems of PDE's obtained, $\mathbf{S}_5$:

$$\begin{aligned}&z_2^3(8f^2f_{z_2z_2z_2}-32ff_{z_2}f_{z_2z_2}+24f_{z_2}^3)+z_2^2(20f^2f_{z_2z_2}-28ff_{z_2}^2-2f^4f_{z_2})\\&\quad+16z_2^{3/2}(f^2g_{z_2}-ff_{z_2}g)+z_2(4f^2f_{z_2}-f^5)g=0,\\&f_{z_1}-2z_2^{1/2}g_{z_2}=0.\end{aligned}$$

Reduction 6 By using the generator $\mathbf{v}_4+\mathbf{v}_f$ we obtain the similarity variables and similarity solutions

$$\begin{aligned}&z_1=x-\textstyle\int t^{-1}\alpha(t)dt, && z_2=z/t,\\&u=f(z_1,z_2), && v=\tfrac{1}{t}(g(z_1,z_2)-f(z_1,z_2)\alpha(t)),\end{aligned}$$

and the systems of PDE's obtained, $\mathbf{S}_6$:

$$\begin{aligned}&4f^2g_{z_1}-4ff_{z_1}g+(-ff_{z_1z_1}+3f_{z_1}^2-ff_{z_1}-f^4)f_{z_2}\\&\quad+f^2f_{z_1z_1z_2}+(f^2-3ff_{z_1})f_{z_1z_2}=0,\\&z_2f_{z_2}+g_{z_1}=0.\end{aligned}$$

4 Invariance analysis of the $(1+1)$-dimensional systems

In several cases, the reduced systems of $(1+1)$-PDE's admit symmetries which lead to further reductions to systems of ODE's, we shall use again the techniques of Lie group theory. The system S_i, $i = 1,\ldots,6$ admits the following symmetries:

$$S_1 : \mathbf{v}_{11} = \frac{\partial}{\partial z_1} - \mu f \frac{\partial}{\partial g}, \quad \mathbf{v}_{12} = -z_1 \frac{\partial}{\partial z_1} + 2z_2 \frac{\partial}{\partial z_2} + f \frac{\partial}{\partial f},$$

$$S_2 : \mathbf{v}_{21} = z_2 \frac{\partial}{\partial z_2}, \quad \mathbf{v}_{22} = 2\frac{\partial}{\partial z_1} - f \frac{\partial}{\partial g}\frac{\partial}{\partial f},$$

$$S_3 : \mathbf{v}_{31} = a\frac{\partial}{\partial z_1} - f\frac{\partial}{\partial g}, \quad \mathbf{v}_{32} = z_1 \frac{\partial}{\partial z_1} - 2z_2 \frac{\partial}{\partial z_2} - f \frac{\partial}{\partial f},$$

$$S_4 : \mathbf{v}_{41} = \frac{\partial}{\partial z_1}, \quad \mathbf{v}_{\beta} = \beta(z_2)\frac{\partial}{\partial z_2} - \beta'(z_2) g \frac{\partial}{\partial g},$$

$$S_5 : \mathbf{v}_{51} = \frac{\partial}{\partial z_1}, \quad \mathbf{v}_{52} = 2z_1 \frac{\partial}{\partial z_1} + 2z_2 \frac{\partial}{\partial z_2} - f\frac{\partial}{\partial f} - 2g\frac{\partial}{\partial g}.$$

1. For system $\mathbf{S_1}$, by using $c\mathbf{v}_{11} + \mathbf{v}_{12}$ we obtain the similarity variable and similarity solutions

   $$w = -\sqrt{z_2}(z_1 - c), \quad f = \sqrt{z_2}h(w), \quad g = -c\mu\sqrt{z_2}h(w) + k(w),$$

   and the system of ODE's

   $$2k_w - wh_w - h = 0,$$

   $$w(h^2 h_{www} - 4hh_w h_{ww} + 3h_w^3 - h^4 h_w) - 8h^2 k_w + 8hh_w k + 2h^2 h_{ww} - 3hh_w^2 - h^5 = 0.$$

2. For system $\mathbf{S_2}$, by using $c\mathbf{v}_{21} + \mathbf{v}_{22}$ we obtain the similarity variable and similarity solutions

   $$w = \frac{e^{cz_1}}{z_2^2}, \quad f = h(w), \quad g = k(w) - \frac{z_1 f}{2},$$

and the system of ODE's

$$ck_w + h_w = 0,$$

$$c^2w^3(h^2h_{www} - 4hh_wh_{ww} + 3h_w^2) + c^2w^2(3h^2h_{ww} - 4hh_w^2) \\ +cw(2h^2k_w - 2hh_wk) - wh^4h_w + c^2h^2h_w - h^3 = 0.$$

3. For system $\mathbf{S_3}$, by using $c\mathbf{v}_{31} + \mathbf{v}_{32}$ we obtain the similarity variable and similarity solutions

$$w = z_2^{1/2}(z_1 + c\mu), \quad f = hz_2^{1/2}, \quad g = ch(w)z_2^{1/2} + k(w),$$

and the system of ODE's

$$\begin{gathered} k_w = 0, \\ w(h^2h_{www} - 4hh_wh_{ww} + 3h_w^3 - h^4h_w) \\ +8h^2k_w - 8hh_wk + 2h^2h_{ww} - 3hh_w^2 - h^5 = 0. \end{gathered} \tag{6}$$

4. For system $\mathbf{S_4}$, by using $c\mathbf{v}_{41} + \mathbf{v}_\beta$ we obtain the similarity variable and similarity solutions

$$w = z_1 - c\int \tfrac{1}{\beta(z_2)}, \quad f = h, \quad g = \tfrac{1}{\beta(z_2)}k(w),$$

and the system of ODE's

$$\begin{gathered} k_w - ch_w = 0, \\ \mu c(4hh_wh_{ww} - h^2h_{www} - 3h_w^3 + h^4h_w) \\ +4h^2k_w - 4hh_wk = 0. \end{gathered} \tag{7}$$

5. For system $\mathbf{S_5}$, by using $c\mathbf{v}_{51} + \mathbf{v}_{52}$ we obtain the similarity variable and similarity solutions

$$w = \tfrac{z_1+c}{z_2}, \quad f = z_2^{-1/2}h(w), \quad g = z_2^{-1}k(w),$$

and the system of ODEs

$$2wk_w + 2k + h_w = 0,$$

$$w^3(4h^2h_{www} - 16hh_wh_{ww} + 12h_w^3) + w^2(12h^2h_{ww} - 16hh_w^2) \\ +w(8h^2k_w - 8hh_wk - h^4h_w + 4h^2h_w) + 4h^2k = 0.$$

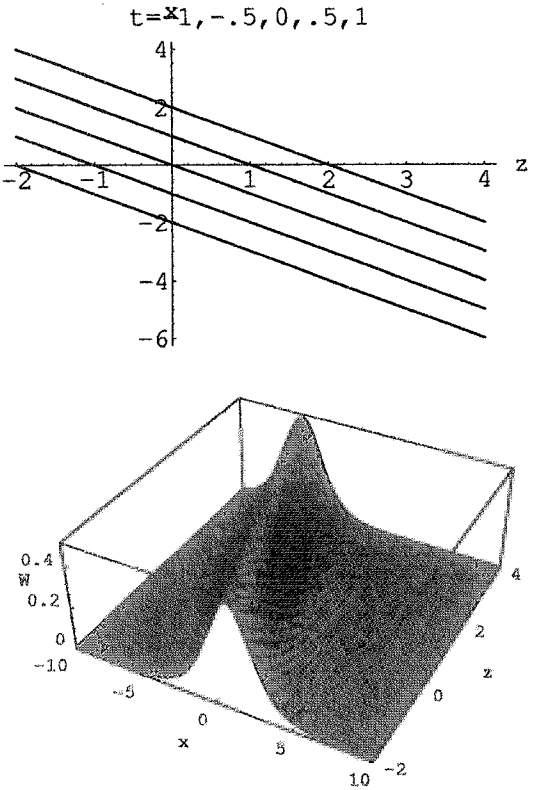

Figure 1. A solution of the SKdV Eq. (1).

5 Some explicit solutions

We can observe that system S_4, also admits an infinite dimensional subalgebra $\mathbf{v}_\beta$ by using $c\mathbf{v}_{41} + \mathbf{v}_\beta$ we obtain (7). From (7), by using the first of these equations we get

$$k = ch + k_1,$$

and the following third order ODE

$$\mu c(h^4 h_w - 3h_w^3 + 4hh_w h_{ww} - h^2 h_{www}) - 4k_1 hh_w = 0.$$

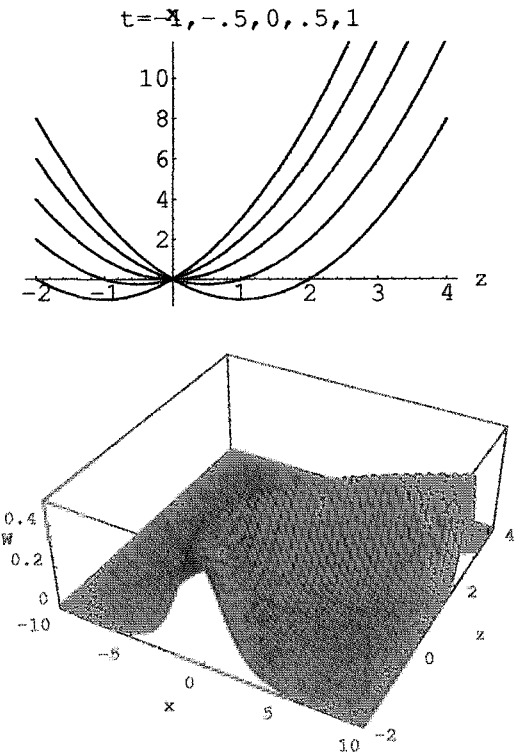

Figure 2. A solution of the SKdV Eq. (1).

By dividing by h^4 and integrating once with respect to w we obtain

$$2k_2h - ck_1(hh_{ww} + h_w^2 + h^4) = 0,$$

whose solution is

$$\int \frac{1}{\sqrt{\frac{2\,h^2\,k_3 - 4\,h\,k_2 + c\,h^4\,k_1}{c\,k_1}}}\,dh = w + k_4.$$

Setting $k_2 = 0$ we obtain the explicit solution

$$h = \frac{2\sqrt{k_3}}{\sqrt{2}\,\sinh\left(\sqrt{2}\sqrt{k_3}\,(w + k4)\right)} \tag{8}$$

The next step will consist of using the above solutions for the ODE's, together with the corresponding symmetry reductions, to construct solutions for the equation (1). From (8), setting $k_3 = \frac{1}{2}$ and $k_4 = 0$, we obtain the following solitonic solution for (1).

$$W = \frac{1}{2}\text{sech}^2\left(\frac{1}{2}(x - c\int \frac{1}{\beta(\mu z - t)} - \frac{1}{\mu}\int \alpha(t))\right). \tag{9}$$

It is interesting that this solitonic solution for the SKdV equation in (2+1) has a very rich structure due to the arbitray functions $\alpha(t)$ $\beta(z_2)$ with $z_2 = \mu z - t$. A plot of solution (9) is given in Figure 1 and Figure 2.

6 Conclusions

In this work we have carried out a detailed Lie symmetry analysis of the $(2+1)$-dimensional integrable generalization of the SKdV equation. Through this invariance analysis we obtain a set of six $(1+1)$-dimensional systems of PDE's. The invariance study of these systems lead to a set of systems of ODE's. From a single soliton solution of the reduced system of ODE's we find that the original equation possesses not only the rich line soliton structure, but also a curve soliton.

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TORI BREAKDOWN IN COUPLED MAP LATTICES

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Abstract of the communication

Coupled Map Lattices (CMLs) are extended dynamical systems with discrete time and space (a lattice), while the field is continuous. Broadly speaking, a local dynamical system is located at each point of the space and interacts, via some spatial coupling, with a set of 'neighbouring' sites. Commonly, the local dynamics has strong chaotic properties.
Nowadays, we have at our disposal a great number of numerical studies on many different models of CML which reveal a rich phenomenology and, in particular, complex spatio-temporal patterns. Moreover, the theoretical analysis of these systems made great strides since the first mathematical papers on CMLs. Hence we know, both from experiment and from theory, that not only simple spatio-temporal patterns (coherent structures) but also spatio-temporal chaos (space-time mixing) can be found in the phase space of these systems. The transition between different phases of the motion is ruled by control parameters, typically expressing the strength of the spatial interaction which couples the sites. It is therefore of interest to clarify the mechanisms (i.e. bifurcations) which make the space-time chaotic phase emerge from the non-chaotic one. To approach this problem it is natural, at the outset, to go back to what we have learned for non-extended systems.
In particular, the role of tori breakdown in the creation of irregular behaviour for such systems is well known. In this talk we presented a numerical study [1] of the behaviour and the breakdown of tori in a lattice of diffusively coupled logistic maps introduced by Kaneko [2]:

$$x_i^{k+1} = f(x_i^k) + \frac{\epsilon}{2}\left(f(x_{i-1}^k) - 2f(x_i^k) + f(x_{i+1}^k)\right),$$

where $i = 1, \ldots, N$ represents the lattice point (i.e. the spatial variable), k is the discrete time and periodic boundary conditions are assumed. The logistic map $f(x) = Rx\,(1-x)$, provides the local dynamics at each site i. Commonly,

R is chosen large enough to provide chaotic behavior. The two parameters ϵ and R are usually referred to as the coupling and the nonlinearity.
In the paper [1] six examples of tori which behave in different ways are discussed. In particular, the Afraimovich and Shilnikov's route [3] to the destruction of one-dimensional phase locked tori is confirmed; moreover, also other mechanisms (crises) occur in the breakdown. They not always lead to chaos and are strictly linked to the peculiarity of this system, i.e. to its "non generic" nature due to the presence of a group of spatial symmetries which commute with the map. In fact, the dynamics of the CML is strongly influenced by the presence of a family of heteroclinic cycles [4, 5, 6], which, in some cases, plays an important role in the destruction of invariant tori.

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EVOLUTION OF THE UNIVERSE IN TWO HIGGS-DOUBLETS STANDARD MODELS

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In Quantum Field Theory models of electro-weak interactions with spontaneously broken gauge invariance, renormalizability limits to four the degree of the Higgs potential, whose minimum determines the vacuum in tree approximation. Through the discussion of some simple variants of the Standard Model with two Higgs doublets, we show that the technical limit imposed by renormalizability may prevent the physical realizability of some phases of the system, that would be otherwise allowed by the symmetry of the Lagrangian of the system. We show that the incorporation into an effective Lagrangian of suitable composite particle fields may resolve this discrepancy.

1 Introduction

This paper will be devoted to a discussion the mathematical foundations for the description of the possible evolution patterns of our Universe, in the framework of simple variants of the Standard Model (SM) of Glashow-Weinberg-Salam [1] for unified weak and electromagnetic (e.m.) interactions of elementary particles. As well known, the SM is a relativistic Quantum Field Theory, with spontaneously broken gauge invariance. The characteristics of the model are assigned by specifying the internal symmetry group G of the formalism, the basic field operators (acting in the Hilbert space of state vectors of the system and representing the elementary particles described by the model) and their transformation properties under the relativistic space-time symmetry group (Poincaré group) and under G transformations. The dynamics is encoded in a Poincaré and G-invariant Lagrangian function of the basic fields, thought of as classical fields. The internal symmetry group G is basically a gauge group $SU_2 \times U_1$, which can be complemented with some discrete transformation group, required by phenomenology. The transformation properties of the basic fields under the group U_1 are specified through the (eigen-) value of the

hypercharge operator Y, which is identified with the generator of infinitesimal U_1 transformations. The basic fields consist in *matter fields*, associated to the quarks and the leptons, *gauge fields*, associated to the vector bosons $W^{\pm}$, Z^0 and to the photon γ (these particles mediate the weak and electromagnetic interactions), and *scalar fields* (with respect to Poincaré transformations) $\phi = (\phi_1, \ldots, \phi_n)$, some of which represent the so called Higgs bosons. All the basic fields are classified according to irreducible representations of G.

The "actual" symmetry group of the theory coincides with the invariance group of the vacuum state (the ground state of the theory). This is identified to the isotropy subgroup of G at the point of absolute minimum of a G-invariant potential $V(\phi)$, which is a polynomial function of the scalar fields. In the determination of the minimum of $V(\phi)$, each component of ϕ has to be considered as a numeric constant in space-time (as we shall do in the following), to be identified to the vacuum expectation value (VEV) $\langle \phi_i \rangle_0$ of the operator valued field ϕ_i. The linear action of the group G in the vector space spanned by ϕ defines a G-space Φ. Being the potential invariant under G, its extrema will be degenerate along G-orbits, so that V is essentially a function in the orbit space Φ/G [2]. Vacuum states associated to minima of the potential lying on the same G-orbit are equivalent, since they can be related by a gauge transformation and the isotropy subgroups of G at points of the same orbit are conjugate in G. All the points of Φ with conjugated isotropy subgroups of G form a *stratum* in Φ, which is formed by the union of one or more orbits.

By Hilbert theorem, the G-invariant polynomial $V(\phi)$ can be expressed as a polynomial $\widehat{V}(p)$ in the elements $p = (p_1, \ldots, p_q)$ of a minimal integrity basis (MIB) of the ring of the G-invariant polynomials in ϕ. The function $\widehat{V}(p)$ yields a suitable analytical expression of the function induced in the orbit space Φ/G by $V(\phi)$. In fact the points of orbit space are conveniently parametrized by the elements of a MIB. The location of the point ϕ_0 of absolute minimum of $V(\phi)$ in Φ (or, equivalently, the point p_0 of absolute minimum of $\widehat{V}(p)$ in Φ/G) and its isotropy subgroup of G (and the conjugacy class of isotropy subgroups of the points lying in the orbit identified by p) obviously depend on the coefficients $a = (a_1, \ldots, a_k)$ of the polynomial $\widehat{V}$, that will be called *phenomenological parameters*, since their values are related to the masses and self coupling constants of the scalar particles. All the points of minimum of $V(\phi)$ (of $\widehat{V}(p)$) lying in a same *stratum* of Φ (of Φ/G) are considered to determine a unique *phase* of the system, and the phase will be considered as *physically realizable* (to have non-zero probability to be seen in nature) if it originates from minima which are stable against small variations

of the phenomenological parameters. A geometrical analysis of the G-space Φ, allows to determine all the strata Σ of Φ (and their images $\widehat{\Sigma}$ in Φ/G) [2] and, therefore, all the possible phases of the system allowed by the symmetry properties of the model.

The only effective method known at present to solve the dynamical equation of the model, which are derived in a standard "classical" way from the Lagrangian, is perturbation theory. In a naïve application of perturbative methods, however, infinities are encountered, originating from ill defined products of operator valued generalized functions, like the quantum fields. A consistent procedure (*renormalisation theory*) exists for cancelling the unwanted divergencies, only if some additional constraint (*renormalizability*) is imposed on the form of the Lagrangian. In particular, renormalizability requires the degree of the polynomial $V(\phi)$ to be ≤ 4 and thismay drastically reduce the number of physicaly realizable phases. In fact, while for general G-invariant polynomial potentials of sufficiently high degree ($\geq$ of the double of the highest degree of the elements of a MIB [3]), for any given stratum Σ there is always a region of non-zero measure $\mathcal{R}_\Sigma$ in the space of the phenomenological parameters, such as, for every $a \in \mathcal{R}_\Sigma$, the absolute minimum of $V(\phi)$ lies in Σ, it may happen that the measure of this region reduces to zero for a lower degree polynomial [3], as, for instance, a *renormalizable* potential of general form. In this case we shall say that the potential is *incomplete*. Now, the assumptions of renormalizability, however important it can be for the perturbative resolubility of the dynamics of the model, is a technical assumption. So, the fact that it may exclude the physical realizability of some of the phases allowed by the symmetry of the model is quite disturbing.

Since our declared aim is to describe all the possible patterns of evolution of our Universe in the set of phases allowed by the symmetry of the Lagrangian, in a first approach to the problem we shall study two variants of the SM with and without the constraint of renormalizability on the corresponding potentials. From this analysis we shall get sufficient hints on how to restore renormalizability, without modifying the set of physically realizable phases. This will be attained through the introduction of suitable additional scalar fields, to be thought of as representing bound states of the original scalar particles, bound states that are created when suitable external conditions are established.

In the SM the spontaneous symmetry breaking of the gauge symmetry $SU_2 \times U_1$ of the Lagrangian is obtained though the introduction of a doublet of scalar fields, that is, two complex scalar fields transforming as the components of a vector of an irreducible 2-dimensonal representation of SU_2, of hypercharge 1. The experimental information about the scalar sector of the

SM is, however, very poor and leaves room to the introduction of other scalar SU_2 singlets or multiplets. There are, in particular quite strong motivations for extensions of the SM with the introduction of at least one additional scalar doublet with hypercharge 1. The extension is required, for instance, if the model is considered as a low energy limit of the minimal supersymmetric model, which necessarily contains two scalar doublets. Other motivations for additional doublets arise in axion models or from models of CP violation or in grand unified theories in order to generate sufficient baryon number [4].

The introduction of an additional scalar doublet in the SM requires also an enlargement of the internal symmetry group by means of a discrete transformation, induced by an involutive operator $\hat{i}$, changing the sign of only one of the two scalar doublets. Invariance under this transformation prevents the coupling of both scalar doublets to the quark fields, that would give rise to unwanted phenomenological consequences (flavor changing neutral current effects).

In the following we shall analyze two possible extensions of the SM along the lines indicated above. In particular, for each model we shall do the following:

1. determine all the strata of the G-space Φ and, therefore all the phases allowed by the symmetry of the model, independently of the choice of the potential;

2. construct a *complete* polynomial invariant potential, *i.e.*, a possibly non renormalizable polynomial invariant potential, which is sufficiently general to make physically realizable all the phases allowed by the symmetry of the model;

3. determine the orbit space of the action of G on the scalar fields and the *phase space* of the physical system described by the model, *i.e.*, identify, in the space of phenomenological parameters, the stability region of each phase and determine the possible second and first order inter-phase boundaries;

4. find the domain of stability of our present day Universe and the possible paths of its evolution.

2 Model 1

The symmetry group of the Lagrangian is $G^{(1)} = SU_2 \times U_1$ and its elements are represented by a pair $(u, e^{i\theta})$, where $u \in SU_2$ and θ is real. The model includes,

Strata	Defining relations	Symmetry
$S^{(4)}$	$p_1 > \sqrt{p_2^2+p_3^2+p_4^2}$	Z_2
$S^{(3)}$	$p_1 = \sqrt{p_2^2+p_3^2+p_4^2}$	$U_1^{e.m.}$
$S^{(0)}$	$p_1 = p_2 = p_3 = p_4 = 0$	$G^{(1)}$

Table 1. The index in the symbols distinguishing the strata denotes their dimension in orbit space. The isotropy group Z_2 is generated by the element $(-\mathbb{1}_2, -1)$ of $SU_2 \times U_1$ and the group $U_1^{e.m.}$ by the elements $(\mathrm{diag}\{e^{i\theta}, e^{-i\theta}\}, e^{i\theta})$.

besides the matter and gauge fields, two doublets φ and χ of hypercharge 1. The model is not realistic, since it violates natural flavor conservation by neutral current effects in the phase that should correspond to the present phase of our Universe, but it is interesting for our purposes, since it shows that renormalizability does not necessarily prevent the realizability of a complete set of symmetry allowed phases..

A MIB is yielded by $p_i = I_i$, $i = 1, \ldots, 4$, where the I_i's are the following four polynomials of second degree:

$$I_1 = \varphi^\dagger\varphi + \chi^\dagger\chi, \quad I_2 = \varphi^\dagger\varphi - \chi^\dagger\chi, \quad I_3 = 2\Re\left[\varphi^\dagger\chi\right], \quad I_4 = 2\Im\left[\varphi^\dagger\chi\right]. \tag{1}$$

The orbit space and the images of the strata of Φ in it can be characterized by means of the $\widehat{P}(p)$ matrix, defined by one of the present authors [2]. This matrix, in the present case, can be easily calculated to be

$$\frac{1}{4}\widehat{P}(p) = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_2 & p_1 & 0 & 0 \\ p_3 & 0 & p_1 & 0 \\ p_4 & 0 & 0 & p_1 \end{pmatrix}, \tag{2}$$

where the p_i's are to be thought of as independent variables in orbit space. ¿From positive semi-definiteness and rank conditions of the matrix $\widehat{P}(p)$ one determines the stratification of orbit space described in Table 1.

In the Euclidean space $\mathbb{R}^4$, with Cartesian coordinates $(p_1, \ldots, p_4)$, the orbit space is the half-cone bounded by the surface of equation $p_1 = \sqrt{\sum_i p_i^2}$. The tip of the cone corresponds to the stratum $S^{(0)}$, and the rest of the surface to the stratum $S^{(3)}$, while the interior points form $S^{(4)}$.

In order to determine the physically realizable phases when the theory is renormalizable, a fourth degree polynomial invariant potential has to be defined

$$\widehat{V}(p) = \frac{1}{2}\sum_{i,j=1}^{4} A_{ij}p_ip_j + \sum_{i=1}^{4} a_ip_i = \frac{1}{2}\sum_{i,j=1}^{4} A_{ij}(p_i-\eta_i)(p_j-\eta_j) - \frac{1}{2}\sum_{i,j=1}^{4} A_{ij}\eta_i\eta_j, \tag{3}$$

where $\eta_i = -\sum_{j=1}^{4}(A^{-1})_{ij}a_j$.

To make sure that $\widehat{V}(p)$ is bounded below, we shall assume that the symmetric real matrix A is positive definite and, only in order to simply the calculations, without modifying the essence of the results, we shall set $A = \mathbb{1}$. Let $C = C^+ \cup C^-$ denote the double cone bounded by the surfaces of equation $\eta_1 = \pm\sqrt{\sum_{i=2}^{4}\eta_i^2}$. It is then clear that for all η's internal to C^+ (to C^-), the absolute minimum of $V(p)$ is stable in $S^{(4)}$ ($S^{(0)}$), while for η's external to C it is stable in $S^{(3)}$. For η on the surface of C^+ (of C^-), the absolute minimum is on $S^{(3)}$ (in $S^{(0)}$), but it is unstable.

If one keeps for the matrix $A > 0$ a general form, the results do not essentially change: The cone is simply rotated and deformed as a consequence of independent rescalings along the coordinate axes. So, in the space of the phenomenological parameters $(a_1, \ldots, a_4)$, there are three disjoint open regions of stability of the three phases allowed by the symmetry of the model and associated to the strata of the orbit space. These three regions are separated by inter-phase boundaries, formed by second order phase transition points.

As explained in ref. [2], in correspondence with stable minima, the number of null eigenvalues of the mass matrix of the scalar bosons (which is equal to the matrix of second order derivatives of V) is equal to the dimension of the orbit hosting the minimum (so, 4, 3 or 0, according as the orbit is in $S^{(4)}$, $S^{(3)}$ or $S^{(0)}$, respectively), that is, to the number of Goldstone bosons. If the minimum is unstable, an excess of zero eigenvalues appears, corresponding to the presence of *pseudo-Goldstone bosons*. The corresponding eigenvectors lie in directions which are orthogonal to the stratum at the point of minimum of V.

The evolution of the Universe described by the model can be represented by a continuous line in the space of the parameters $(a_1, \ldots, a_4)$. A random path in this space has zero probability of crossing the origin, so the only physically realizable second order phase transitions correspond to transitions between the phases associated to the strata $S^{(4)}$ and $S^{(3)}$ or $S^{(3)}$ and $S^{(0)}$.

We can conclude that this model is both renormalizable and *complete*.

3 Model 2

The model contains the same fields as Model 1, but the symmetry group of the Lagrangian is assumed to be $G^{(2)} = \mathrm{SU}_2 \times \mathrm{U}_1 \times \{\hat{i}\} \times \{U_{\mathrm{CP}}\}$, where $\hat{i}$ is the generator of a Z_2 reflection group and U_{CP} is the (anti-unitary) generator of CP-transformations. This group is considered by many authors as a possible alternative to the SM group for the description of the present phase of our Universe. At the end of this papers we shall confute the objections moved by some authors agains this model. The transformation properties of the scalar doublets φ and χ under the additional discrete symmetries are the following:

$$\hat{i}: \ (\varphi, \chi) \to (\varphi, -\chi); \qquad U_{\mathrm{CP}}: \ (\varphi, \chi) \to (\varphi^*, \chi^*). \tag{4}$$

and a MIB is yielded by the following four polynomials of second and fourth degree:

$$p_1 = I_1, \qquad p_2 = I_2, \qquad p_3 = I_3^2, \qquad p_4 = I_4^2, \tag{5}$$

where the I_i's are defined in (1). The associated $\widehat{P}(p)$ matrix has the following form

$$\frac{1}{4}\widehat{P}(p) = \begin{pmatrix} p_1 & p_2 & 2p_3 & 2p_4 \\ p_2 & p_1 & 0 & 0 \\ 2p_3 & 0 & 4p_1p_3 & 0 \\ 2p_4 & 0 & 0 & 4p_1p_4 \end{pmatrix}. \tag{6}$$

The stratification of orbit space, obtained from the matrix $\widehat{P}(p)$ just defined, are described in Table 2.

Let us now momentarily forget the renormalizability condition. Then, a simple potential that is already sufficient to describe all the variants of our Universe evolution allowed by the symmetry of Model 2, is the following:

$$\widehat{V}_2(p) = \sum_{i,j=1}^{4} b_i p_i^2 + a_i p_i, \tag{7}$$

where the coefficients are all real. It is not difficult to realize that the potential just defined admits stable minima in each of the strata listed in Table 2, for suitable values of the parameters $a_1, \ldots, a_4$. We have also determined the stability regions in the space of the a_i's for each phase allowed by the symmetry. The results are resumed in Fig. 1, where each phase is represented

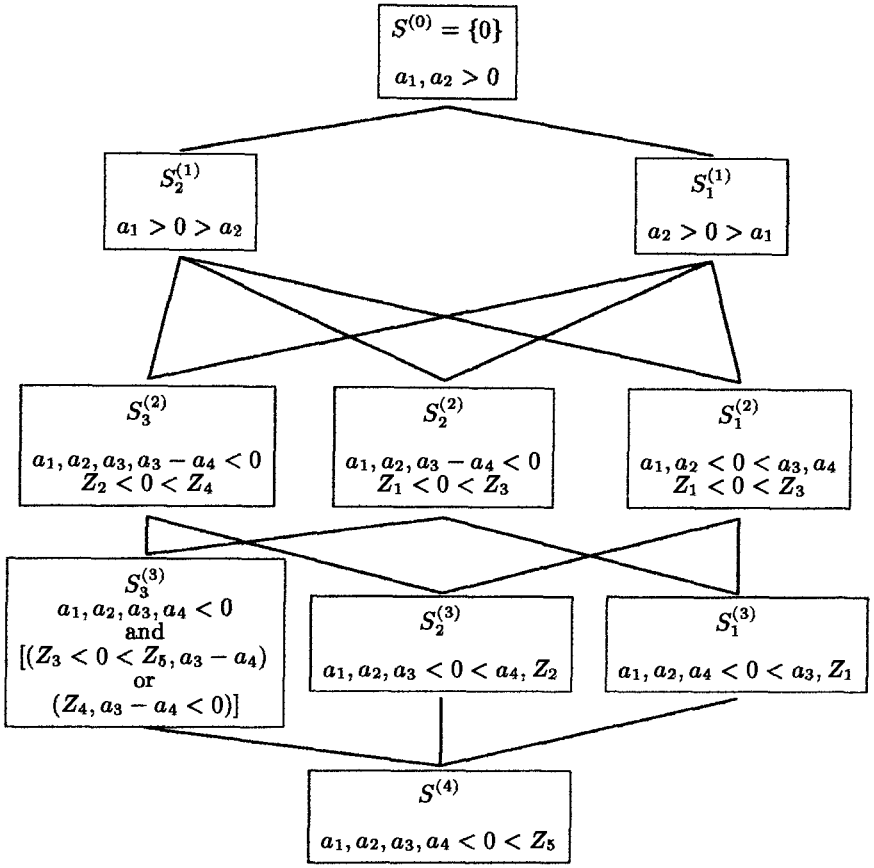

Figure 1. Possible phases and corresponding stability regions in the space of the phenomenological parameters $a_1, \ldots, a_{12}$. Possible second order phase transitions are only allowed between bordering strata, connected, in the figure, by continuous lines. The following expressions of the phenomenological parameters are used: $Z_1 = a_1a_2+a_4/2$; $Z_2 = a_1a_2+a_3/2$; $Z_3 = (a_1-2a_2a_3)(a_2-2a_2a_3)/(1-4a_3^2)^2-(a_3-a_4))/8$; $Z_4 = (a_1 - 2a_2a_4)(a_2 - 2a_1a_4)/(1 - 4a_4^2)^2 - (a_4 - a_3))/8$; $Z_5 = -16a_1^2a_2^2 + a_3^2 + a_4^2$.

Stratum	Defining relations	Symmetry
$S^{(4)}$	$p_1, p_3, p_4, p_1^2 - p_2^2 - p_3 - p_4 > 0$	$\{\mathbb{1}\}$
$S_1^{(3)}$	$p_3 = 0 < p_4 < p_1^2 - p_2^2$	$\{\mathrm{U}_1^{\mathrm{e.m.}}(\pi)\,\hat{i}\,U_{CP}\}$
$S_2^{(3)}$	$p_4 = 0 < p_3 < p_1^2 - p_2^2$	$\{U_{CP}\}$
$S_3^{(3)}$	$p_1^2 - p_2^2 - p_3 - p_4 = 0 > p_1, p_3, p_4$	$\mathrm{U}_1^{\mathrm{e.m.}}$
$S_1^{(2)}$	$p_3 = p_4 = 0 < p_1, p_1^2 - p_2^2$	$\{\mathrm{U}_1^{\mathrm{e.m.}}(\pi)\,\hat{i}\} \times \{U_{CP}\}$
$S_2^{(2)}$	$p_1^2 - p_2^2 - p_4 = p_3 = 0 < p_1, p_4$	$\mathrm{U}_1^{\mathrm{e.m.}} \times \{\hat{i}\,U_{CP}\}$
$S_3^{(2)}$	$p_1^2 - p_2^2 - p_3 = p_4 = 0 < p_1, p_3$	$\mathrm{U}_1^{\mathrm{e.m.}} \times \{U_{CP}\}$
$S_1^{(1)}$	$p_1 - p_2 = p_3 = p_4 = 0 < p_1$	$\mathrm{U}_1^{\mathrm{e.m.}} \times \{\mathrm{U}_1^{\mathrm{e.m.}}(\pi)\,\hat{i}\} \times \{U_{CP}\}$
$S_2^{(1)}$	$p_3 = p_4 = 0 < p_1 = -p_2$	$\mathrm{U}_1^{\mathrm{e.m.}}\,\hat{i}\} \times \{U_{CP}\}$
$S^{(0)}$	$p_1 = p_2 = p_3 = p_4 = 0$	$G^{(2)}$

Table 2. The upper index in the symbols labeling the strata denotes the dimension of the stratum in orbit space, while the lower one distinguishes distinct strata with the same dimension. The group $\mathrm{U}_1^{\mathrm{e.m.}}$ is the subgroup of $\mathrm{SU}_2 \times \mathrm{U}_1$ formed by the elements $(\mathrm{diag}\{e^{i\theta}, e^{-i\theta}\}, e^{i\theta})$ and $\mathrm{U}_1^{\mathrm{e.m.}}(\pi)$ denotes its element $(\mathrm{diag}\{-1, 1\}, -1)$.

by a box in which the relevant stability conditions are specified. The group-subgroup correlations between nearby strata (phases) are indicated by straight lines. The 3-dimensional inter-phase boundaries turn out to be related to second order phase transitions.

Let us assume that Model 2, governed by the potential (7), correctly formalizes the properties of our present day Universe. In this case, the most likely values of the phenomenological parameters will correspond to a point lying in the stability region of the phase associated to the stratum $S_3^{(3)}$, with symmetry $\mathrm{U}_1^{\mathrm{e.m.}}$. This point has to be close to the boundary with the phase associated to the stratum $S_1^{(2)}$, with symmetry $\mathrm{U}_1^{\mathrm{e.m.}\hat{\mathrm{i}}} \times \{U_{CP}\}$, owing to the weakness of CP-violation, justified by the smallness of the expression Z_4, defined in the caption of Figure 2. ¿From this state, the Universe can evolve through the phases indicated in Figure 2, following the straight lines in a random way, without dramatic events (catastrophes).

Let now revert to Model 2 in a renormalizable version, that we shall call Model 3.

$S^{(4)}$	$S_1^{(3)}$	$S_2^{(3)}$	$S_3^{(3)}$	$S_1^{(2)}$	$S_2^{(2)}$	$S_3^{(2)}$	$S_1^{(1)}$	$S_2^{(1)}$	$S^{(0)}$
0	0	0	$2\lambda p_1$	0	$2\lambda_1 p_1$	$2\lambda_1 p_1$	0	λ_1	λ_1
0	0	0	$-2\lambda p_2$	0	$-2\lambda_1 p_2$	$-2\lambda_1 p_2$	λ_1	0	λ_2
0	λ	0	$-\lambda$	λ_1	λ_2	$-\lambda_1$	λ_2	λ_2	λ_3
0	0	λ	$-\lambda$	λ_2	$-\lambda_1$	λ_2	λ_3	λ_3	λ_4

Table 3. Right hand sides of the first four equations (9) for each stratum.

4 Model 3

The internal symmetry group of Model 3 and its content in fields is the same as in Model 2 (so we can choose the same MIB defined in (5)), but for the potential we shall assume the most general fourth degree invariant polynomial, to make sure that no possible phase is lost:

$$\widehat{V}_3(p) = \frac{1}{2}\sum_{i,j=1}^{2} A_{ij}\, p_i\, p_j + \sum_{i=1}^{4} a_i\, p_i, \tag{8}$$

where all the parameters are real and $A_{12} = A_{21}$.

The extremum condition for $\widehat{V}(p)$ can be conveniently written in the orbit space, where they assume the following form in a stratum $\widehat{\Sigma}$, defined by the relations $\hat{f}_\alpha(p) = 0,\ \alpha \in \mathcal{I}_{\hat{\Sigma}}$ and $\hat{f}_\beta(p) > 0,\ \beta \in \mathcal{I}_{\hat{\Sigma}}$, where $\mathcal{E}_{\hat{\Sigma}}$ and $\mathcal{I}_{\hat{\Sigma}}$ are the set of values of of the indices α and β involved in the equalities and, respectively, inequalities, defining the stratum $\widehat{\Sigma}$ in orbit space [2]:

$$\frac{\partial \widehat{P}(p)}{\partial p_i} = \sum_{\alpha \in \mathcal{E}_{\hat{\Sigma}}} \lambda_\alpha \frac{\partial \hat{f}_\alpha(p)}{\partial p_i}, \tag{9}$$

where the λ_α's are real Lagrange multipliers and, for each stratum, the $\hat{f}_\alpha$'s and $\hat{f}_\beta$'s can be read out of Table 3. The left and right hand sides of the first four equations in (9) can be collected in two four dimensional vectors. The first one has the following components:

$$(A_{11}p_1 + A_{12}p_2 + a_1,\ A_{12}p_1 + A_{22}p_2 + a_2,\ a_3,\ a_4), \tag{10}$$

while the form of the second one depends on the stratum and is listed in the columns of Table 3, for each stratum of orbit space.

It is immediate to realize, by means of the first four equations in (9) and Table 3, that there can be extrema of the potential in the strata of

dimension ≥ 3 only if there exsist some relations between the values of the a_i's ($a_3 = a_4 = 0$, $a_4 = 0$, $a_3 = 0$ and $a_3 = a_4$, respectively, for the strata $S^{(4)}$, $S_1^{(3)}$, $S_2^{(3)}$ and $S_3^{(3)}$). These conditions reduce to zero the measures of the regions of stability of these extrema in the space of phenomenological parameters. So, there will not be stable phases associated to the strata of dimension ≥ 3.

We can conclude that this model is renormalizable, but it is *incomplete.* In tree approximation, the phases associated to the strata of dimension ≥ 3 have zero probability to be observed in nature and the statement holds true at present, for the past and the future.

A considerable effort has been made to determine the radiative corrections to the potential at higher perturbative orders and finite temperature contributions. None of these corrections can increase the initial number of phenomenological parameters in the resulting effective potential, and they must not reduce the initial symmetry of the fourth degree polynomial potential. Of course, the corrections can shift the localization of the absolute minimum of the potential [5], but no new phase can be created and no unstable phase can gain stability [6].

5 Completing Model 3: Model 4

The comparison of Model 2 and Model 3 suggests a manner to bypass the incompatibility between *renormalizability* and *completeness.* In fact, if some of the basic SU_2 invariant polynomials appearing in the terms of degree ≥ 4 of the potential V, defined in (7), are treated as independent scalar fields, describing bound states of the original scalar particles, the potential, written in terms of the new fields, can take on a renormalizable form. In order to maintain both renormalizability and completeness, the choice of the composite fields has to fulfill some conditions:

i) The symmetry of the composite fields must be higher than the symmetry of the initial scalar fields (the isotropy subgroup in $G^{(2)}$ of the new fields has to be larger than the isotropy group, Z_2, of the pair (φ, χ)). In the particular case of Model 2, the isospin of the composite particles has to be zero. This is a consequence of the fact that we are not interested in finding new possible phases that could originate from non zero VEV's of the new fields. We shall rather consider these VEV's as induced by non zero VEV's of the initial basic fields. To realize this situation, the coefficient of the squares of the composite fields in the potential will have to be positive.

Fields	SU_2	U_1	$\hat{i}$	CP
φ	$\underline{2}$	1	1	φ^*
χ	$\underline{2}$	1	-1	χ^*
F	$\underline{1} \oplus \underline{1}$	2	-1	F^*
G	$\underline{1} \oplus \underline{1}$	2	-1	$-G^*$
H	$\underline{1}$	0	-1	H
L	$\underline{1}$	0	-1	$-L$

Table 4. Full set of fields for Model 4 and corresponding transformation properties under the group $G^{(2)}$.

ii) The role of the composite particles is to mimic the effect of the higher degree terms that make the potential complete. So, in the potential there must be terms which are linear in the composite fields and quadratic in the initial basic fields φ and χ.

The requirements i) and ii) can be satisfied in a unique way, described in Table 4, where the complete set of fields to be considered as basic fields in the construction of the Lagrangian of Model 4 is listed, with the corresponding transformation properties under the internal symmetry group $G^{(2)}$.

A MIB, constructed with the fields listed in Table 4, contains more than 30 elements and would be too long to be written down here. We shall limit ourselves to define a slightly simplified form of a *complete* fourth degree polynomial invariant potential, which, nevertheless, is sufficient to show that all the phases listed in Table 2 possess a non-zero measure stability region in the space of the phenomenological parameters:

$$\begin{aligned}\widehat{V}_4 = &\textstyle\sum_{i=1}^{4} \left(a_i\, I_i + b_i\, I_i^2\right) + a_5\, H^2 + b_5\, H^4 + a_6\, L^2 + b_6\, L^4 + a_7\, |F|^2 + b_7\, |F|^4 \\ &+ a_8\, |G|^2 + b_8\, |G|^4 + a_9\, H\, I_3 + a_{10}\, L\, I_4 + 2\, a_{11}\, \Im\left[I_5\, F^*\right] + 2 a_{12}\, \Re\left[I_5\, G^*\right]\end{aligned} \tag{11}$$

where the invariants I_i, $i = 1, \ldots, 4$ are defined in (1) and $I_5 = \left[\chi^T \tau_2 \varphi\right]$ is a complex second degree SU_2-invariant polynomial. In V_4, the coefficients b_i's and $a_5, \ldots, a_8$ will be considered to be positive and it will not be essentially restrictive for our goals, to put all the b_i's equal to 1.

It will be worth noticing that 33 independent phenomenological parameters are required to write down the most general fourth degree invariant polynomial in the fields listed in Table 5. Due to the fact that four of them (b_5, b_6, b_7 and b_8) must not affect the phase diagram, since a_5, a_6, a_7 and a_8 are assumed to be positive, we attain a number of 29 essential phenomenolog-

ical parameters. In order to assure the physical realizability of all the phases allowed by the symmetry of the Lagrangian, the minimal degree of the potential must not be less than 8 [3]. The number of free parameters involved in the definition of a general eighth degree polynomial, depending on the MIB (5), is also 29. Thus, the number of parameters defining the phase diegram relative to Model 2 and Model 4, with a general form for the potential, is exactly the same.

The polynomial (11) may be thought of as a suitable approximation of the general form of a fourth degree polynomial, built in terms of all the fields of the model. Its use is suggested by the fact that it allows an explicit analytical determination of all the regions of stability in the space (a_1, a_2), for each of the phases listed in Table 2. Let us consider, for example, the phase corresponding to the stratum $S_3^{(3)}$. In order to write down the extremum conditions for the potential (11) relevant to this phase, let us introduce the following general parametrization of the fields of the model:

$$\varphi = \begin{pmatrix} r_1\, e^{i\alpha_1} \\ r_2\, e^{i\alpha_2} \end{pmatrix}; \qquad \chi = \begin{pmatrix} q_1\, e^{i\beta_1} \\ q_2\, e^{i\beta_2} \end{pmatrix}; \qquad F = f\, e^{i\delta_1}; \qquad G = g\, e^{i\delta_2}. \tag{12}$$

and, at points of the stratum $S_3^{(3)}$, $r_1, r_2, q_2, f, g > 0 = q_1 = f = g \neq H\,L$. It will also be useful to define

$$\theta_1 = \alpha_1 + \beta_2 - \delta_1; \;\; \theta_2 = \alpha_2 + \beta_1 - \delta_1; \;\; \theta_3 = \alpha_1 + \beta_2 - \delta_2; \;\; \theta_4 = \alpha_2 + \beta_1 - \delta_2. \tag{13}$$

The solution of the extremum conditions becomes easier in a particular "gauge". By means of a $\mathrm{SU}_2 \times \mathrm{U}_1$ transformation one can always choose $r_1 = \alpha_1 = \beta_1 = \beta_2 = 0$ and in this "gauge" the extremum conditions take the following simplified form:

$$\begin{aligned}
&2r_2 \frac{\partial V_4}{\partial I_1} + 8a_3 r_2^2 q_2 \cos^2\alpha_2 + 8a_4 r_2 q_2^2 \sin^2\alpha_2 + 2a_7 H q_2 \cos\alpha_2 = 0, \\
&2q_2 \frac{\partial V_4}{\partial I_2} + 8a_3 r_2^2 q_2 \cos^2\alpha_2 + 8a_4 r_2^2 q_2 \sin^2\alpha_2 + 2a_7 H r_2 \cos\alpha_2 = 0, \\
&8(a_3 - a_4) r_2^2 q_2^2 \sin\alpha_2 + 2a_7 H r_2 q_2 \sin\alpha_2 = 0, \\
&2H(a_6 + 2b_6 H^2) + 2a_7 r_2 q_2 \cos\alpha_2 = 0.
\end{aligned} \tag{14}$$

If H is mall enough (which is the existence condition of a second order phase transition), from the last two equations one obtains $H \approx (a_7/a_6) r_2 q_2 \cos\alpha_2$,

with $\cos\alpha_2 = a_7/(a_6(a_3 - a_4))$. After insertion of these values of H and $\cos\alpha_2$, the first two equations become linear in r_2^2 and q_2^2 and can be easily solved for these variables.

6 Conclusions

The aim of the analysis performed on some simple examples was to illustrate the fact that, if the renormalizability of the theory has to be considered as an actual physical constraint on any model of quantum gauge field theory (as was done in the case of an extension of the SM model with two Higgs doublets), additional scalar particles have to be included in the theory. The number and type of such particles depends on the initial fields defining the properties of our Universe and on the discrete subgroup that is added to gauge group $SU_2 \times U_1$, to complete the specification of the symmetry of the Lagrangian of the model. If the composite particles are not included, the model can only predict an truncated history of the Universe.

The claims by Branco, on the necessity of three Higgs doublets in order to break spontaneously CP, are not justified in the framework of Model 4, due to the fact that the phase $S_3^{(3)}$ is physically realizable

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Possible ground states of D-wave condensates in isotropic space through geometric invariant theory

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A complete and rigorous determination of the possible ground states for D-wave pairing Bose condensates is presented. Using an orbit space approach to the problem, we find 15 allowed phases (besides the unbroken one), with different symmetries, that we thoroughly determine, specifying the group-subgroup relations between bordering phases.

1 Introduction

The problem of classifying Spontaneous Symmetry Breaking (S.S.B.) patterns in theories where the ground state is determined as a minimum of a potential invariant under the action of a compact group G of transformations is relevant both in elementary particle physics and in solid state physics.

Even if trivial in principle, the concrete determination of minima of G-invariant potentials is, generally, a difficult task, owing to the degeneracies of the extremal points. A geometric approach, based on the analysis of local properties of the G-spaces, has been devised to exploit the invariance properties of the potential. For many years, the study of the lattice of the G-space isotropy subgroups, complemented with the famous Michel's conjecture, was used to determine the residual symmetry after S.S.B. Independently, in 1971, during

the first years of the development of the G-space approach, Gufan proposed the use of a fundamental system of polynomial invariants (integrity bases) to write the most general form of Landau non-equilibrium potential [1]. But it was in 1981, when counter-examples to Michel's conjecture began to be discovered, that a new rigorous method, fully exploiting geometric invariant theory, was proposed [2]. It was demonstrated that the range of a set of basic polynomial invariants yields an isomorphic image $\overline{\mathcal{S}}$ of the orbit space. A clear method (hereafter called the *P-matrix* or *orbit space* approach), founded on a sounder mathematical analysis, was discovered to construct $\overline{\mathcal{S}}$.

In solid state physics, the analytical precision of the approach seemed to be not essential to treat second order phase transitions. Actually, the P-matrix approach appeared to be complicated in real cases, for high cardinality and/or high degree integrity bases. Some results were obtained with numerical techniques, artificially truncating the non-equilibrium Landau potential in order that just the lower degree invariants appeared in the expansion [3,4].

In the following years, it was also proved that the P-matrix allows to get, at least in principle, a model independent classification of the theoretically admissible S.S.B. schemes [5].

This communication aims at demonstrating that the geometric method may be successfully employed in a real problem: the study of the possible ground states of a D-wave condensate.

After a brief account of the physical motivations behind such problem, we shall exhibit the results and some mathematical achievements that have permitted to give a detailed description of the orbit space structure of the problem. Finally, we shall make a remark, showing that the geometrical structure of the orbit space is sufficient to study the S.S.B. mechanism.

2 Physical motivations

Superfluidity and superconductivity are justified on the basis of the macroscopic condensation of Bose quasi-particles. The classical Bardeen, Cooper and Schrieffer theory for superconductivity dates 1957. Soon after, a BCS-type transition was proposed for the Fermi system ^{3}He by Anderson and Morel [6]. Cooper pair formation was thought to occur in an $L \neq 0$ state, to take into account the hard core nature of ^{3}He atoms interaction. The superfluid phases were actually observed [7], and the nature of p-wave pairing is now well established for ^{3}He. The theory of $L \neq 0$-superfluids is also relevant for "unconventional" superconductivity. The high temperature superconducting (HTS) oxides are anomalous in their non-Fermi liquid normal state properties and share with heavy fermion superconductors unconventional d-wave pairing [8,9].

The underlying microscopic mechanism inducing superconductivity in these materials is still unclear and is one of today's major challenges [10,11].

Such a situation has motivated the efforts at studying the macroscopic properties of unconventional superconductors through the Landau theory of phase transitions [12]. Moreover, the identification of the order parameter symmetry may be considered as a preliminary task in the construction of viable models describing the attractive nature of the pairing interaction.

The possible ground states of a high L superfluid has been the object of intense investigations during the 60's and the 70's [6,13,14,15].

A definitive answer to the classification of possible symmetry breaking patterns in D-wave pairing Bose condensate, in the framework of the Landau theory of phase transitions was recently given in [16].

3 The orbit space approach

Let us briefly recall some basic elements of the orbit space approach [17]. To this end, we shall denote by $x \in \mathbf{R}^n$ a vector order parameter, transforming linearly and orthogonally under the compact real symmetry group G, and by $\Phi(\alpha; x)$ the G-invariant free energy, expressed also in terms of state variables α. The points $x_0(\alpha)$, where the function $\phi_\alpha(x) = \Phi(\alpha; x)$ takes on its absolute minimum, correspond to the stable phase of the system, whose symmetry is determined by the isotropy subgroup, G_{x_0}, of G at x_0. Owing to G-invariance, the stationary points of the free energy are degenerate along G-orbits. Since the isotropy subgroups of G at points of the same orbit are conjugate in G, only the conjugate class, $[G_{x_0}]$, of G_{x_0} in G, *i.e.* the *symmetry* (or *orbit-type*) of the orbit through x_0, is physically relevant.

The set of all G-orbits, endowed with the quotient topology and differentiable structure, forms the *orbit space*, $\mathbf{R}^n/G$, of G and the subset of all the G-orbits with a given symmetry forms a *stratum* of $\mathbf{R}^n/G$. Phase transitions take place when, by varying the values of the α's, the point $x_0(\alpha)$ is shifted to an orbit lying on a different stratum. If $\Phi(\alpha; x)$ is a sufficiently general function of the α's, by varying these parameters, it is possible to shift $x_0(\alpha)$ on whichever stratum of $\mathbf{R}^n/G$. So, the strata are in one-to-one correspondence with the symmetry phases allowed by the G-invariance of the free energy. On the contrary, extra restrictions on the form of the free energy function, not coming from G-symmetry requirements (e.g., the assumption that the free energy is a polynomial of low degree), can limit the number of allowed phases.

Being constant along each G-orbit, the free energy may be conveniently thought of as a function defined in the orbit space of G. This fact can be formalized using some basic results of invariant theory. In fact, the G-invariant

polynomial functions separate the G-orbits, meaning that, for any two distinct G-orbits, there is at least a polynomial G-invariant function assuming different values on them. Moreover, every G-invariant polynomials can be built as real polynomial functions of a finite set, $\{p_1(x), \ldots, p_q(x)\}$, of basic polynomial invariants (*integrity basis of the ring of G-invariant polynomials*), which need not, for general compact groups, be algebraically independent. The number of algebraically independent elements in a em minimal set of basic polynomial invariants is $n - \nu$, where ν is the dimension of the generic (principal) orbits of G. Information on the number and degrees of a minimal set of basic invariants, and the degrees of the algebraic relations (*syzygies*) among them, can be inferred from the Mölien function of G.

Let us call q_0 the number of independent elements of the set $\{p\}$. The range of the *orbit map*, $x \mapsto p(x) = (p_1(x), \ldots, p_q(x)) \in \mathbf{R}^q$, yields a realization of the orbit space of the linear group G, as a connected semi-algebraic surface, *i.e.* a subset of $\mathbf{R}^q$, determined by algebraic equations and inequalities. The orbit space of G is, therefore, a closed and connected region of a q_0-dimensional algebraic surface, delimited by lower dimensional semi-algebraic surfaces.

Like all semi-algebraic sets, the orbit space of G presents a natural *stratification*, since it can be considered as the disjoint union of semi-algebraic subsets of various dimensions (*geometrical strata*), each stratum being in the border of a higher dimensional stratum, but for the highest dimensional one, which is unique (*principal stratum*). The connected components of the *symmetry strata* are in one-to-one correspondence with the *geometrical strata*. The symmetries of two bordering strata are related by a group subgroup relation (up to conjugacy) and the lower dimensional stratum has a larger symmetry.

The orbit space can be identified with the semi-algebraic variety, S, formed by the points $p \in \mathbf{R}^q$, satisfying the following conditions i) and ii)[2,18]:

i) p lies on the surface, Z, defined by the syzygies;

ii) the $q \times q$ matrix $\widehat{P}(p)$, defined by the relation

$$\widehat{P}_{ab}(p(x)) = \sum_{j=1}^{n} \partial_j p_a(x)\, \partial_j p_b(x), \quad \forall x \in \mathbf{R}^n \tag{1}$$

is positive semidefinite at p.

The relations defining the strata can be obtained as positivity and rank conditions on the matrix $\widehat{P}(p)$ and the minimum of $\Phi(\alpha; x)$ can be computed as a constrained minimum in orbit space of the function $\widehat{\Phi}(\alpha; p)$,

$$\widehat{\Phi}(\alpha; p(x)) = \Phi(\alpha; x), \ \forall x \in \mathbf{R}^n \tag{2}$$

or from the solutions of the equation[2]

$$\sum_{b=1}^{q} \widehat{P}_{ab}(p)\, \partial_b \widehat{\Phi}(\alpha; p) = 0, \qquad a = 1, \ldots, q, \tag{3}$$

which is equivalent to the state equation, $\partial\Phi(\alpha; x)/\partial x_j = 0$, $j = 1, \ldots, n$.

4 Symmetry of the allowed D-wave condensate states in isotropic space

The formation of D-wave condensate states breaks the symmetry of the isotropic 3-dimensional space, which corresponds to the group $\mathbf{O}_3 \otimes \mathbf{U}_1 \times \langle \mathcal{T} \rangle$, where $\mathbf{O}_3$ is the complete rotation group, $\mathbf{U}_1$ is the group of gauge transformations and $\langle \mathcal{T} \rangle$ is the group generated by the time reversal operator $\mathcal{T}$.

The symmetry of the allowed D-wave condensate ground states is defined by the relative values of the complex coefficients in the decomposition of the gap-function, Δ, in terms of spherical harmonics with $L = 2$:

$$\Delta(\theta, \phi) = \sum_{m=-2}^{2} D_m\, Y_2^m(\theta, \phi) \tag{4}$$

The set of functions $\{Y_2^m, Y_2^{m\,*}\}$ yields a basis of a ten-dimensional (10 D) space hosting a real representation of the symmetry group $\mathbf{O}_3 \otimes \mathbf{U}_1 \times \langle \mathcal{T} \rangle$. A general element, γ, of the group will be denoted by a triple $\gamma = (\rho, e^{i\phi}, \epsilon)$, where, $\rho \in \mathbf{O}_3$, $0 \leq \phi < 2\pi$ and $\epsilon = -1$, or $+1$ according as a time reflection is involved in the transformation, or not.

The action of G can be transferred to a real irreducible action on the 10 D vector formed by the coefficients $\{D_2, \ldots, D_{-2}, D_{-2}^*, \ldots, D_2^*\}$. The representation of G thus obtained can be realized in the 10 D real vector space of a couple of two independent, real, second rank, symmetric, traceless tensors, $X_{ij}^{(1)}$ and $X_{ij}^{(2)}$, $i, j = 1, 2, 3$, which can be considered as the real and imaginary parts of a complex 3×3 matrix ψ, whose elements will be written in terms of five complex coordinates, z_j:

$$z_j = x_j + i\, x_{5+j}, \qquad j = 1, \ldots, 5; \qquad x_i \in \mathbf{R}, \tag{5}$$

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} z_2 + \frac{z_5}{\sqrt{3}} & z_1 & z_3 \\ z_1 & -z_2 + \frac{z_5}{\sqrt{3}} & z_2 \\ z_3 & z_2 & -\frac{2\,z_5}{\sqrt{3}} \end{pmatrix}. \tag{6}$$

The matrix ψ transforms in the following way under a general transformation $\gamma = (\rho, \phi, \epsilon) \in G$:

$$\gamma \cdot \psi = e^{i\phi}\, \rho\, \psi'\, \rho^{\mathrm{T}}, \qquad \gamma \in G, \tag{7}$$

where $\psi' = \psi$ or ψ^*, according as $\epsilon = +1$ or -1 and the apex T denotes transposition. As a consequence, the group G acts as a group of linear, real, *orthogonal* transformations on the vector order parameter $x \in \mathbf{R}^{10}$.

The kernel of the representation of G just defined is the group generated by the space reflection. So, it will not be restrictive to assume that the symmetry group is $G = \mathbf{SO}_3 \otimes \mathbf{U}_1 \times \langle \mathcal{T} \rangle$ and, when speaking of G, in the following, we shall always refer to this linear group acting in the vector space $\mathbf{R}^{10}$.

The linear group G has a trivial principal isotropy subgroup (the isotropy subgroup of generic points of $\mathbf{R}^{10}$), thus the principal G-orbits have the same dimensions as G and its *orbit space*, *i.e.*, the quotient space $\mathbf{R}^{10}/G$, has dimensions $q_0 = 10 - 4 = 6$.

The Mölien function of G, $M(\eta)$, can be calculated in the form of an invariant Haar integral over G (see, for instance,[19,20]):

$$M(\eta) = \int_G \frac{d\mu(g)}{\det(\mathbf{1} - \eta\, \mathrm{g})}, \qquad |\eta| < 1, \tag{8}$$

where $\mu(g)$ is a normalized invariant measure on the group G, the integration is over the whole group G and $g \in G$. An explicit calculation of the integral leads to

$$M(\eta) = \frac{\eta^{20} + \eta^{12} + \eta^{10} + \eta^8 + 1}{(1-\eta^2)(1-\eta^4)^2(1-\eta^6)^2(1-\eta^8)}. \tag{9}$$

Equation (9) yields the following indications, whose validity has been checked through direct calculations:

1. A minimal integrity basis for the linear group G contains nine elements, $\{p_1, \ldots, p_9\}$ with degrees $(d_1, \ldots, d_9) = (2, 4, 4, 6, 6, 8, 8, 10, 12)$.

2. The invariants p_i are connected by five independent syzygies of degrees 16, 18, 20, 22 and 24.

3. The most general G-invariant polynomial, like a general non-equilibrium polynomial Landau potential, $\widehat{\Phi}(\alpha; p)$, can be written as a polynomial function of the elements of the integrity basis $\{p_i\}_{i=1\ldots9}$, in terms of five arbitrary polynomials[21], $Q_i = Q_i(\alpha; p_1, \ldots, p_6)$, $i = 0, \ldots, 4$:

$$\widehat{\Phi} = Q_0 + Q_1\, p_7 + Q_2\, p_8 + Q_3\, p_9 + Q_4\, p_7\, p_9. \tag{10}$$

The elements of the minimal integrity basis can be chosen in the following form:

$$\begin{array}{lll} p_1 = \mathrm{Tr}(\psi\psi^*) = \sum_{i=1}^{10} x_i^2, & p_6 = \Re\left[\mathrm{Tr}(\psi^2)\mathrm{Tr}(\psi^2\psi^*)\mathrm{Tr}(\psi^{*3})\right], \\ p_2 = \mathrm{Tr}\left[(\psi\psi^*)^2\right], & p_7 = \Re\left[\mathrm{Tr}(\psi^{*2})\left(\mathrm{Tr}(\psi^2\psi^*)\right)^2\right], \\ p_3 = \left|\mathrm{Tr}(\psi^2)\right|^2, & p_8 = \Re\left[\mathrm{Tr}(\psi^2\psi^*)\mathrm{Tr}(\psi^3)\left(\mathrm{Tr}(\psi^{*2})\right)^2\right], \\ p_4 = \left|\mathrm{Tr}(\psi^3)\right|^2, & p_9 = \Re\left[\left(\mathrm{Tr}(\psi^2)\right)^3\left(\mathrm{Tr}(\psi^{*3})\right)^2\right]. \\ p_5 = \left|\mathrm{Tr}(\psi^2\psi^*)\right|^2, & \end{array} \tag{11}$$

Using these definitions, we have determined the explicit form of the syzygies, of the $\widehat{P}$-matrix elements, of the equations and inequalities determining the strata in the orbit space of G. For each stratum, denoted by $S^{(d,r)}$, where d denotes the dimension and r is an enumeration index, we have picked up a "typical point" and determined the corresponding isotropy subgroup of G. All these results are essential pre-requisites for a rigorous calculation of the minima of a G-invariant polynomial along the lines indicated in the Introduction.

Part of our results are resumed in Tables 1, 2 and 3 and in Fig. 1, where the group–subgroup relations among the symmetries of bordering strata are also specified.

The explicit expressions of the syzygies and of the elements of the $\widehat{P}(p)$ matrix and the relations defining higher dimensional strata would require too much space to be written down here. We just stress that to overcome some computational difficulties in the full characterization of the stratification of the orbit space, we have devised a sophisticated procedure, allowing us to obtain rational parametric equations also for the higher dimensional strata. The procedure is quite general and is reminiscent, but in a different context and dimensionality, of classical techniques for parameterizing plane algebraic curves.

As a simple example of the effectiveness of our approach, we have calculated the minimum of a general fourth degree polynomial G-invariant free energy

$$\widehat{\Phi}^{(4)}(p) = \alpha_0 \frac{p_1^2}{2} + \sum_{j=1}^{3} \alpha_j p_j, \quad \alpha_i \in \mathbf{R} \tag{12}$$

in the additional assumptions that it is bounded below and has a local maximum at the origin ($\alpha_1 < 0$); obviously, in (12) the α_i, $i = 1 \ldots 3$ are phenomenological parameters.

With the definitions:

$$\tilde{p}_i = \frac{p_i}{p_1^{d_i/2}}, \qquad \tilde{p} = (\tilde{p}_2, \ldots, \tilde{p}_9), \tag{13}$$

and

$$\Delta(\tilde{p}) = \alpha_0 + 2\,\alpha_2\,\tilde{p}_2 + 2\,\alpha_3\,\tilde{p}_3, \tag{14}$$

the polynomial $\widehat{\Phi}^{(4)}(p)$ can be put in the following convenient form:

$$\widehat{\Phi}^{(4)}(p) = \frac{p_1^2}{2}\,\Delta(\tilde{p}) + \alpha_1\,p_1. \tag{15}$$

Since, owing to its definition, p_1 ranges over the whole non negative real numbers, $\widehat{\Phi}^{(4)}(p)\big|_{p\in S}$ is bounded below, in the assumption $\alpha_1 < 0$ (*stability condition*), if and only if the minimum, δ, of $\Delta(\tilde{p})\big|_{p_1=1}$ is positive. Being the minimum of the r.h.s. of (15), thought of as a function only of $p_1 \geq 0$, equal to $-\alpha_1/(2\Delta)$, the absolute minimum of $\widehat{\Phi}^{(4)}(p)\big|_{p\in S}$ is $-\alpha_1^2/(2\delta)$. In this way, the problems of rendering explicit the stability condition and evaluating the minimum of $\widehat{\Phi}(p)\big|_{p\in S}$ are reduced to the calculation of δ.

The absolute minimum of δ in each singular stratum can be easily obtained from the equations of the strata, listed in Tables 1 and 2 for strata with dimensions < 3. For the principal stratum, it is easier to solve the projection of equation (3) in the unit sphere ($\sum_{i=1}^{10} x_i^2 = p_1 = 1$), which, using (13), can be written in the form [5]

$$\sum_{j=2}^{9} P_{ij}(\tilde{p})\,\frac{\partial \Delta(\tilde{p})}{\partial \tilde{p}_j} = 0 \tag{16}$$

and to select, subsequently, the solutions lying in the principal stratum.

A comparison of the values of the minima in the different strata, obtained in this way, leads to the results resumed in Table 4 and illustrated in Figure 2. Owing to the low degree of the polynomial defining $\widehat{\Phi}^{(4)}(p)$ in (12), and the consequent low number of free parameters α, the absolute minimum presents

Table 1: Relations defining strata, $S^{(1,r)}$, of dimensions 1 in orbit space. For $2 \leq i \leq 9$, $q_i = p_i/(p_1^{d_i/2})$ and d_i denotes the degree of the polynomial $p_i(x)$ (x is the order parameter).

r	$\tilde{p}_2$	$\tilde{p}_3$	$\tilde{p}_4$	$\tilde{p}_5$	$\tilde{p}_6$	$\tilde{p}_7$	$\tilde{p}_8$	$\tilde{p}_9$
1	1	0	0	0	0	0	0	0
2	1/2	1	1/6	1/6	1/6	1/6	1/6	1/6
3	1/2	1	0	0	0	0	0	0
4	1/3	0	1/3	0	0	0	0	0
5	1/2	0	0	0	0	0	0	0

Table 2: Relations defining strata, $S^{(2,r)}$, of dimensions 2 in orbit space. The $\tilde{p}_i$'s are defined as in Table 1 and $\epsilon = \pm 1$.

$\tilde{p}\backslash r$	1	2	3	4	5
$\tilde{p}_2$	$(2+\xi^2)/6$	$1/2$	$(2+\xi^2)/6$	$1/2$	ξ
$\tilde{p}_3$	0	ξ	ξ^2	1	$2-2\xi$
$\tilde{p}_4$	$(2-\xi)^2(1+\xi)/12$	0	$(2-\xi)^2(1+\xi)/12$	ξ	0
$\tilde{p}_5$	0	0	$\xi^2(1+\xi)/12$	ξ	0
$\tilde{p}_6$	0	0	$(2-\xi)(1+\xi)\,\xi^2/12$	ξ	0
$\tilde{p}_7$	0	0	$(1+\xi)\,\xi^3/12$	ξ	0
$\tilde{p}_8$	0	0	$(2-\xi)(1+\xi)\,\xi^3/12$	ξ	0
$\tilde{p}_9$	0	0	$(2-\xi)^2(1+\xi)\,\xi^3/12$	ξ	0
ξ range	$]0, \frac{1+3\epsilon}{2}[$	$]0,1[$	$]0,\epsilon[$	$]0, \frac{1}{6}[$	$]\frac{1}{2}, 1[$

Table 3: Possible symmetry strata, $S^{(d,r)}$, for D-wave driven pairing in isotropic **space** (d denotes the dimension of the stratum and r is a secondary enumeration index). From left to right, the columns refer to phase reference numbers according to our((d,r)) classification, order ($|H|$) and residual symmetry group (H), complex coordinates ($z_j = x_j + i x_{5+j}$, $j = 1,\ldots,5$) of the H-invariant superconducting vector order parameter In column four, the t's are real coordinates, while the v's are complex ones. The notations are the same as in [22], $R_z(\phi)$ denotes a clockwise proper rotation of the vectors, about the z-axis and $\mathbf{O}_2^z = \{\mathrm{R}_z(\phi)\}_{0\le\phi<2\pi} \cup \{C_{2x}\mathrm{R}_z(\phi)\}_{0\le\phi<2\pi}$.

(d,r)	$\lvert H\rvert$	H	$(z_1,\ldots,z_5)$
$(1,1)$	∞	$\langle C_{2x}\mathcal{T}\rangle \times \{\mathrm{R}_z(\phi)\,\mathrm{U}_1(2\phi)\}_\phi$	$(it,-t,0,0,0)$
$(1,2)$	∞	$\mathbf{O}_2^z \otimes \langle\mathcal{T}\rangle$	$(0,0,0,0,t)$
$(1,3)$	16	$\langle C_{2x},\mathcal{T},C_{4z}\mathrm{U}_1(\pi)\rangle$	$(0,t,0,0,0)$
$(1,4)$	24	$\langle C_{2x},C_{2a}\mathcal{T},C_{3\delta}\,\mathrm{U}_1(4\pi/3)\rangle$	$(0,-it,0,0,t)$
$(1,5)$	∞	$\langle C_{2x}\mathcal{T}\rangle \times \{\mathrm{R}_z(\phi)\,\mathrm{U}_1(-\phi)\}_\phi$	$(0,0,t,-it,0)$
$(2,1)$	6	$\langle C_{2x}\mathcal{T},C_{3z}\mathrm{U}_1(4\pi/3)\rangle$	$(it_1,-t_1,t_2,-it_2,0)$
$(2,2)$	4	$\langle C_{2x}\mathcal{T},C_{2z}\mathrm{U}_1(\pi)\rangle$	$(0,0,t_1,it_2,0)$
$(2,3)$	8	$\langle C_{2x},C_{4z}\mathcal{T}\rangle$	$(0,it_1,0,0,t_2)$
$(2,4)$	8	$\langle C_{2x},C_{2z},\mathcal{T}\rangle$	$(0,t_1,0,0,t_2)$
$(2,5)$	8	$\langle C_{2x}\mathcal{T},C_{4z}\mathrm{U}_1(\pi)\rangle$	$(it_1,t_2,0,0,0)$
$(3,1)$	4	$\langle C_{2z},C_{2x}\mathcal{T}\rangle$	$(it_1,t_2,0,0,t_3)$
$(3,2)$	4	$\langle C_{2x},C_{2z}\rangle$	$(0,v_1,0,0,v_2)$
$(4,1)$	2	$\langle C_{2x}\mathcal{T}\rangle$	(it_1,t_2,t_3,it_4,t_5)
$(4,2)$	2	$\langle C_{2z}\rangle$	$(v_1,v_2,0,0,v_3)$
$(6,1)$	1	$\{1\}$	(v_1,v_2,v_3,v_4,v_5)

Table 4: Absolute minimum, $\widehat{\Phi}^{(4)}_{\min} = -\alpha_1^2/(2\delta)$, of a general, bounded below, G-invariant 4-th degree polynomial, $\widehat{\Phi}^{(4)}(\alpha,p) = \alpha_0 p_1^2/2 + \sum_{j=1}^{3}\alpha_j p_j$, $\alpha_1 < 0$, and hosting strata, $S^{(d,r)}$, as functions of the coefficients α. The denomination of the strata is the same as in Table 3.

α range	δ	(d,r)
$\mathcal{R}_1$: $\mathrm{Max}(0,-3\alpha_0/2,-6\alpha_3)<\alpha_2$	$\alpha_0+2\alpha_2/3$	(1,4)
$\mathcal{R}_2$: $-6\alpha_3>\alpha_2>\mathrm{Max}(-\alpha_0-2\alpha_3,2\alpha_3)$	$\alpha_0+\alpha_2+2\alpha_3$	(1,2), (1,3), (2,4)
$\mathcal{R}_3$: $-\alpha_0/2<\alpha_2<\mathrm{Min}(0,2\alpha_3)$	$\alpha_0+2\alpha_2$	(1,1)
$\mathcal{R}_{13}$: $0=\alpha_2<\mathrm{Min}(\alpha_0,\alpha_3)$	α_0	(1,1), (1,4), (1,5), (2,1), (3,1), (4,1)
$\mathcal{R}_{12}$: $\mathrm{Max}(-3\alpha_0/2,0)<\alpha_2=-6\alpha_3$	$\alpha_0-4\alpha_3$	(1,2), (1,3), (1,4), (2,3), (2,4), (3,2)
$\mathcal{R}_{23}$: $-\alpha_0/2<\alpha_2=2\alpha_3<0$	$\alpha_0+4\alpha_3$	(1,1), (1,2), (1,3), (2,4), (2,5)
$\mathcal{R}_{123}$: $\alpha_2=\alpha_3=0<\alpha_0$	α_0	all, except (0,1)

Figure 1: Possible phase transitions between bordering strata, connected, in the figure, by continuous sequences of one or more arrows. The notations are the same as in Table 3.

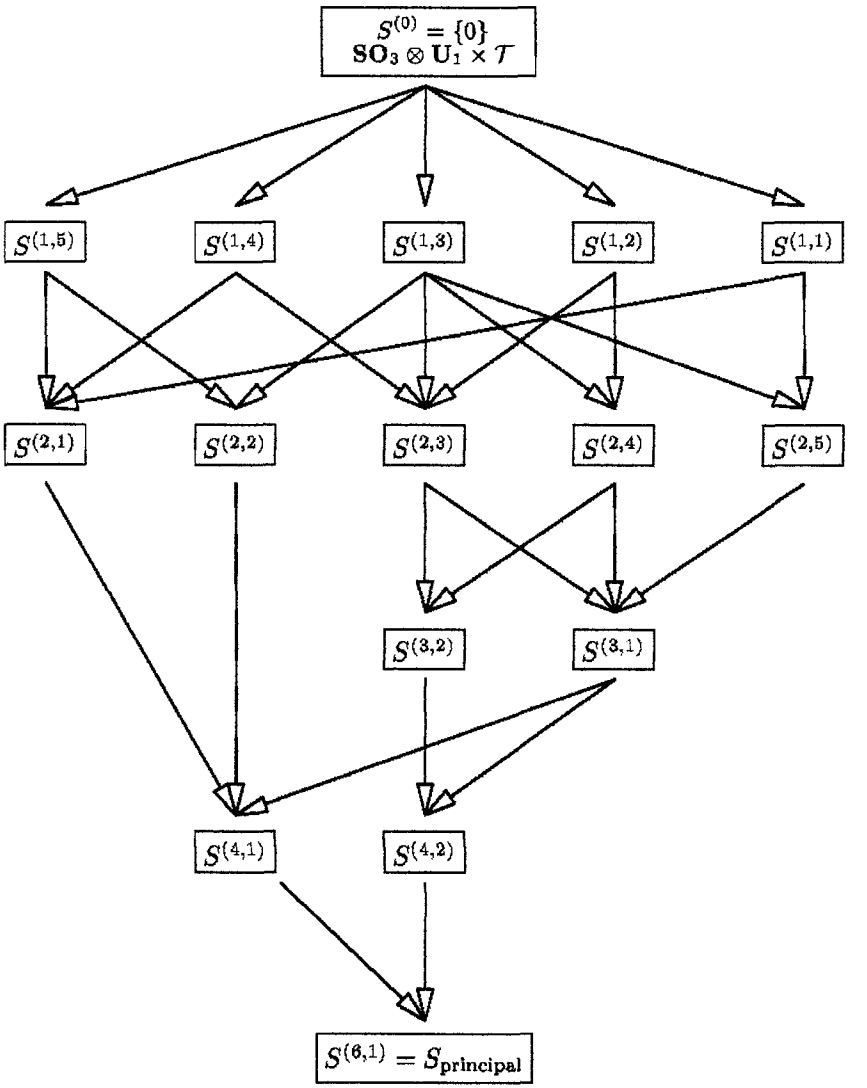
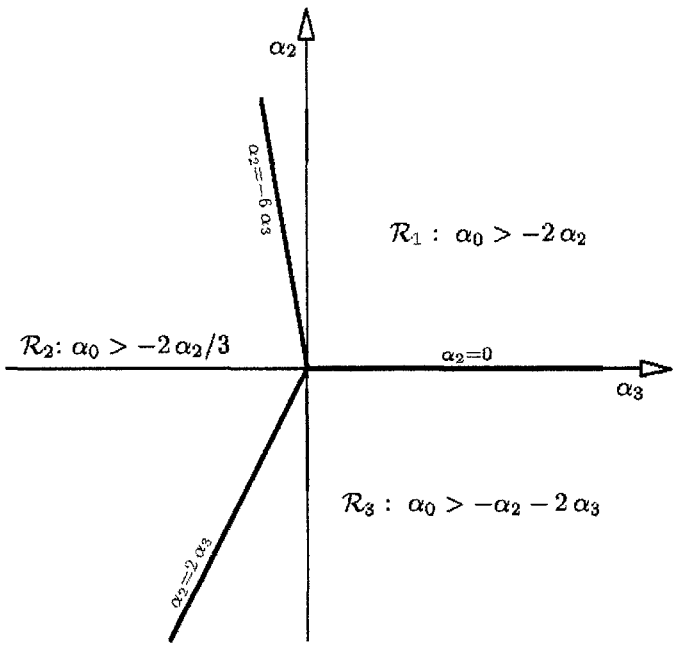


Figure 2: Localization of the absolute minimum of a fourth degree G-invariant polynomial, $\widehat{\Phi}^{(4)}(\alpha, p) = \alpha_0 p_1^2/2 + \sum_{j=1}^{3} \alpha_j p_j$, $\alpha_1 < 0$, as a function of its coefficients. For values of $(\alpha_0, \alpha_2, \alpha_3)$ in $\mathcal{R}_1$ or in $\mathcal{R}_2$ or in $\mathcal{R}_3$, the absolute minimum lies, respectively, in the strata $S^{(1,4)}$ or $\{S^{(1,2)}, S^{(1,3)}, S^{(2,4)}\}$ (degenerate minimum) or $S^{(1,1)}$. For particular values of the α's, see Table 4.



strong degeneracy, particularly for special values of the α's. If these special values are excluded, spontaneous breaking of the symmetry can generate only five distinct phases out of fifteen permitted by the G-symmetry; some of them are unstable. For no non trivial values of (α_2, α_3) does the absolute minimum lie on the stratum $S^{(2,2)}$.

Let us add a few words about the perturbative stability of the three degenerate phases in the region $\mathcal{R}_2$ (see Table 4). For $(\alpha_0, \alpha_2, \alpha_3) \in \mathcal{R}_2$, the addition to the free energy, $\Phi^{(4)}(p)$, of a "small" perturbation, consisting in an invariant polynomial of degree six, $\Theta^{(6)} = \alpha_4\, p_4 + \alpha_5\, p_5$, splits the three degenerate minima determined by the 4-th degreee term[a]. This is easy to check, at least in the additional assumption that the perturbation leaves the absolute minimum in one of the strata corresponding to the degenerate phases. In fact, at the first perturbative order, one obtains from Tables 1 and 2 the following shifts, $\Theta^{(6)}_{(d,r)}$, in the values of the 6-th order free energy at the points where $\Phi^{(4)}(p)$ takes on its degenerate absolute minimum under consideration:

$$\Theta^{(6)}_{(1,2)} = \left(\frac{\alpha_1}{\delta}\right)^3 \frac{\alpha_4 + \alpha_5}{6}, \qquad \Theta^{(6)}_{(1,3)} = 0, \qquad \Theta^{(6)}_{(2,4)} = \left(\frac{\alpha_1}{\delta}\right)^3 (\alpha_4 + \alpha_5)\, \xi, \tag{17}$$

where $0 < \xi < 1/6$.

Since $-\alpha_1/\delta > 0$, the absolute minimum will be perturbatively stable on $S^{(1,2)}$ or, respectively, $S^{(1,3)}$, according as $\alpha_4 + \alpha_5$ is negative or positive.

The difficulties mentioned above can be overcome if one puts less restrictive upper limits to the degree of the polynomial describing the free energy. It is trivial, for instance, to realize that the following class of bounded below polynomial functions [17] have a vanishing maximum at the origin of $\mathbf{R}^{10}$ and display an absolute minimum at the arbitrarily chosen point $\bar{p} \in S$:

$$\sum_{i=1}^{9} \alpha_i \left[(p_i - \bar{p}_i)^{2n_i} - \bar{p}_i^{2n_i} \right], \tag{18}$$

where the α's are positive constants and the n's are positive integers.

Thus, the very fact that, with a convenient choice of the invariant potential, anyone of the phases of the system may be forced to correspond to the ground state, demonstrates that the orbit space formulation offers a big advantage to deal with S.S.B.: it is possible, at least in principle, to determine all the phases of the problem allowed by the considered symmetry, independently of the particular form assumed for the potential. What one really needs to study

[a] The inclusion of terms of 6-th degree, depending only on p_1, p_2 and p_3 would be useless for splitting the degenerate minima, so these terms will be neglected, with no loss of generality.

is just the geometric structure of the orbit space for the physical system under investigation.

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PARENT PHASE AS A ZERO APPROXIMATION IN PHASE TRANSITION THEORY

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The concept of parent phase in phase transition theory and the advantages of this approach are elucidated by consideration of phase transition sequence in $PrAlO_3$.

1 The concept of parent phase

Achievements of phase transition (PT) theory in elements and simple compounds are mainly due to their actual symmetry, which is high enough under definite external conditions. The symmetry provides selection rules and rigorous correlations between interatomic couplings responsible for phase stability. At the same time primitive cells of complex compounds are constituted by several sublattices, which gradually reduce the crystal symmetry being shifted from symmetrical sites due to chemical or any other reasons. The energy responsible for these shifts of the sublattices is of the same order of magnitude as the energy responsible for the PT's. This fact makes it possible to introduce the concept of parent phase. The parent phase is a hypothetical (or actual) structure that roughly coincides with an actual crystal structure, but all the sublattices are arranged at the most symmetrical positions. This approach allows considering a real transition between actual phases as a transition between the phases, whose symmetry groups are subgroups of their joint parent phase symmetry group. It provides additional approximate selection rules, which simplify a mathematical representation of the theory and lead to solutions of some problems, for example, the definition of the energy spectrum of compounds and its dependence on external conditions.

To illustrate the fruitfulness of the parent phase approach we consider a relatively simple example of PT's in the binary oxide $PrAlO_3$, which was of great interest due to its multiphase phase diagram, sufficient chemical stability and possible application of EPR spectra of Pr^{3+} ions in frequency tunable lasers.

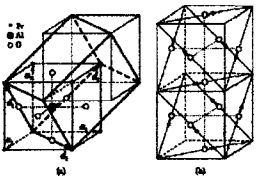

Figure 1. (a) Correlation between the primitive cells of parent cubic phase and actual rhombohedral phase of D_{3d}^6 symmetry. Elementary translations of the cubic (a_1, a_2, a_3) and rhombohedral (d_1, d_2, d_3) phases are shown. (b) Shifts of oxygen ions in the rhombohedral phase with respect to two adjacent primitive cells of the parent phase.

2 Actual crystal structures of $PrAlO_3$

$PrAlO_3$ belongs to a rich family of perovskitelike crystals. It displays D_{3d}^6 symmetry at room temperature [1,2] and undergoes three PT's on cooling down to 50 K [3,4,5] under normal pressure. The set of symmetries corresponding to different phases of $PrAlO_3$ may be arranged in a sequence as follows:

$$D_{3d}^6 \overset{205K}{\longleftrightarrow} D_{2h}^{28} \overset{151K}{\longleftrightarrow} C_{2h}^3 \overset{\approx 90K}{\longleftrightarrow} D_{4h}^{18}, \tag{1}$$

where the values over the arrows are the temperatures of the PT's.

The primitive cell of the rhombohedral phase with D_{3d}^6 symmetry contains two chemical units ($PrAlO_3$). Nodes of the sublattices occupied by ions of different elements are disposed at regular point systems or Wyckoff positions (WP) [6] as shown on Figure 1. The coordinates of two Pr ions are $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and $(\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$ with respect to rhombohedral axes, thus, they occupy WP 2(a). Two Al ions occupy the sites with coordinates $(0, 0, 0)$ and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, *i.e.* WP 2(b). Six oxygen ions belong to the unique sublattice constituted by WP 6(e). Here we present only one point of the 6(e) sublattice $(\frac{3}{4} - x, \frac{3}{4} + x, \frac{1}{4})$ to stress that the point symmetry of the oxygen position is C_2, which allows a temperature dependent shift x along the twofold axis.

The structure of the orthorhombic (D_{2h}^{28}) phase is characterized by an elementary cell that contains four chemical units. Twelve oxygen ions belong

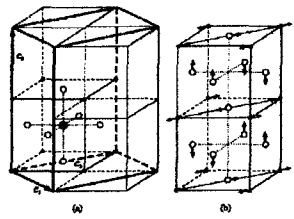

Figure 2. (a) Correlation between the elementary cells of cubic parent phase and actual orthorhombic phase of D_{2h}^{28} symmetry. Elementary translations of the orthorhombic (c_1, c_2, c_3) phase are shown. (b) Shifts of oxygen and Pr ions in the orthorhombic phase with respect to two adjacent primitive cells of the parent phase.

to two WP's (Fig. 2): *i)* 8(g), which can be generated from the representative point $(\frac{1}{4}, y, \frac{1}{4})$, and *ii)* 4(*e*), whose representative point is $(0, \frac{1}{4}, z)$. Both these positions are characterized by variable shift parameters depending on external conditions (temperature, pressure, *etc*). We choose the origin of coordinates at a site proper for Al ion as it was done for the rhombohedral phase. The sublattice of Al ions is situated at WP 4(a), and the Pr ions are arranged at WP 4(*e*) with representative point $(0, \frac{1}{4}, \frac{1}{2}+\delta)$. The symmetry of this position is C_{2v}.

3 Minimal number of phenomenological parameters accounting for the 205 K phase transition

It is obvious that the symmetries of the rhombohedral and orthorhombic phases are not connected by group - subgroup relation. Thus, the PT at $205K$ must be of first order. To describe it, one needs to calculate free energies of the two phases and compare them to determine the line of PT on the temperature - pressure phase diagram. The internal energy of sublattices depends on interatomic interactions, which are very complex to be calculated using a quantum mechanical approach. Usually only isotropic pair interactions in few coordination spheres are assumed. The minimal number of phenomenological

parameters defining a pair interaction is two [7]. Three coordination spheres should be taken into account to describe the perovskite structure stable with respect to fluctuations of composition [8]. The internal energy depends on the interatomic distances, which are defined by the lattice constants and the parameters describing ion shifts from their symmetrical points.

Therefore, it is easy to determine the minimal set of phenomenological parameters describing the PT at $205K$. There are six model potentials of interatomic interactions: Pr-Pr, Pr-O, Pr-Al, Al-Al, Al-O and O-O. They introduce 12 phenomenological parameters in the theory. Interatomic distances in the rhombohedral phase depend on three variables (lattice constant, rhombohedral angle and oxygen shifts). In the orthorhombic phase, interatomic distances depend on 6 variables. Totally, the number of parameters in this very simple "molecular" model of PT in $PrAlO_3$ is 21. Of course, each of these parameters depends on external conditions. Just these dependences are responsible for the PT.

4 Parent phase approximation

The most symmetrical structure known for perovskite compounds is the cubic one with symmetry O_h^1. In this structure, containing one chemical unit, Al ions occupy WP 1(a), while Pr ions occupy WP 1(b) and oxygen ions - WP 3(c). This structure is usually used as a parent phase for PTs in perovskites. In particular, the whole set of structures that are proper for the sequence of PTs (1) can be considered as induced from the parent phase by one irreducible three component order parameter (OP). Its transformation properties under the action of the translation subgroup of O_h^1 are described by the vector of reciprocal space that ends in the R-point of the Brillouin zone of the simple cubic lattice ($\mathbf{k} = \frac{1}{2}(\mathbf{b_1} + \mathbf{b_2} + \mathbf{b_3})$) [9]. It means that the nonequilibrium Landau potential, which accounts for the whole set of PT's (1), should depend on the three invariants:

$$J_1 = \eta_1^2 + \eta_2^2 + \eta_3^2, \quad J_2 = \eta_1^2\eta_2^2 + \eta_1^2\eta_3^2 + \eta_2^2\eta_3^2, \quad J_3 = \eta_1^2\eta_2^2\eta_3^2. \qquad (2)$$

Here η_1, η_2, η_3 - are components of the OP. The possible low symmetry phases in invariant form can be determined from semi-positivity and rank conditions of the Sartori's P-matrix [10,11]:

$$P(J_1, J_2, J_3) = \begin{Vmatrix} 4J_1 & 8J_2 & 12J_3 \\ 8J_2 & 4J_1J_2 + 12J_3 & 8J_1J_3 \\ 12J_3 & 8J_1J_3 & 4J_2J_3 \end{Vmatrix} \qquad (3)$$

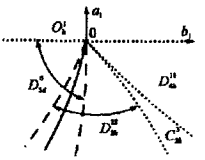

Figure 3. Phase diagram of $PrAlO_3$ near the 5-phase point. The olid line is the line of first order rhombohedral - orthorhombic phase transition. The dotted lines correspond to second order phase transitions. The dashed lines are stability boundaries of rhombohedral and orthorhombic phases.

It follows from (3) that there are 8 phases of different structures, which can be induced by this OP. To describe the four phases of different symmetries that are presented in the sequence (1) it is enough to use a Landau potential of eighth degree with respect to the components of the OP and to assume that the OP is small enough to justify N-phase point approximation.

Below we present the phase diagram obtained within the framework of the Landau potential:

$$\Phi = a_1 J_1 + a_2 J_1^2 + a_3 J_1^3 + a_4 J_1^4 + b_1 J_2 + b_2 J_2^2 + c_{12} J_1 J_2 + c_{112} J_1^2 J_2 + d_1 J_3 + c_{13} J_1 J_3. \tag{4}$$

The Landau potential (4) depends on 10 phenomenological parameters. According to catastrophe theory, only 6 of them may depend on external conditions [12]. It provides the possibility to describe boundaries between phases from the sequence (1). The phase stability boundaries and lines of PT's are qualitatively drawn on the plane (a_1, b_1) (Fig. 3). A phase diagram of this type corresponds to the additional assumptions: $a_2 > 0, d_1 > 0$ and $4a_2 b_2 - c_{12}^2 > 0$.

The PT between the rhombohedral and orthorhombic phases corresponds to the line

$$b_1 = \frac{(9c_{12} + 2d_1)a_1}{9a_2} \tag{5}$$

The region of stability of the rhombohedral phase extends as far as the line

$$b_1 = \frac{(3c_{12}+d_1)a_1}{6a_2} - \frac{(2b_2+c_{13}+3c_{112})a_1^2}{12a_2^2} + \frac{(27a_3+3c_{12}-d_1)(3c_{12}+d_1)a_1^2}{6^3a_2^3} \tag{6}$$

The orthorhombic phase is stable between the lines

$$b_1 = \frac{(2c_{12}+d_1)a_1}{4a_2} - \frac{(b_2+c_{13}+2c_{112})a_1^2}{8a_2^2} + \frac{(12a_3+c_{12}-d_1)(2c_{12}+d_1)a_1^2}{64a_2^3} \tag{7}$$

and

$$b_1 = \frac{c_{12}a_1}{2a_2} - \frac{(b_2+2c_{112})a_1^2}{8a_2^2} + \frac{(12a_3+c_{12})c_{12}a_1^2}{32a_2^3}. \tag{8}$$

At the line (8) a second order PT to the monoclinic C_{2h}^3 phase occurs. The second order PT between the monoclinic and tetragonal (D_{4h}^{18}) phases corresponds to the line

$$b_1 = \frac{c_{12}a_1}{2a_2} - \frac{c_{112}a_1^2}{4a_2^2} + \frac{3a_3c_{12}a_1^2}{8a_2^3}. \tag{9}$$

5 Correlation between the EPR lines of the rhombohedral and orthorhombic phases

Parent phase not only serves as a tool for definition of phase stability regions but also helps to calculate the EPR spectra of rare earth elements in different phases even if their symmetries are not connected by a group - subgroup relation. Thus, according to the parent phase symmetry O_h^1, its subgroups D_{3d}^6 and D_{2h}^{28} are connected by their joint subgroup C_{2h}^6. In the D_{3d}^6 phase the OP components are equal: $\eta_1 = \eta_2 = \eta_3$. In the orthorhombic phase: $\eta_1 = \eta_2, \eta_3 = 0$. Their joint subgroup C_{2h}^6 corresponds to the relation $\eta_1 = \eta_2 > \eta_3$.

We consider the electronic energy level of Pr^{3+} ions that is triply degenerate in the cubic parent phase. It is possible due to cubic point symmetry of WP 1(a) of O_h^1 occupied by Pr ions. In first approximation splitting of the triply degenerate electronic level under the influence of crystal field induced by the OP components is defined by the secular equation:

$$\varepsilon^3 + \varepsilon\left[\tfrac{3}{2}A^2(J_1^2 - 3J_2) - \tfrac{1}{2}B(J_1^2 - J_2)\right] + (3A - B)^2(3A + 2B)J_3 + \tfrac{1}{2}A(B^2 - 5A^2)J_1^3 + \tfrac{1}{2}A[(3A)^2 - B^2]J_1 J_2 = 0, \quad (10)$$

where A and B are diagonal and off-diagonal matrix elements of the crystal field energy, averaged over electronic wave functions in zero approximation, and ε is the value of the splitting. Eq. (10) can easily be solved if $\eta_1 = \eta_2 \neq \eta_3$:

$$\begin{aligned} \varepsilon_{1,2} &= \tfrac{1}{2}\left\{A(\eta_1^2 - \eta_3^2) - B\eta_1^2 \pm \left[(3A(\eta_1^2 - \eta_3^2) + B\eta_1^2)^2 + 8B\eta_1^2\eta_3^2\right]^{1/2}\right\}, \\ \varepsilon_3 &= -A(\eta_1^2 - \eta_3^2) + B\eta_1^2. \end{aligned} \quad (11)$$

Equations (11) connect the electronic spectrum lines in rhombohedral and orthorhombic phases if one supposes $\eta_1^2 = \eta_3^2$ for the former case and $\eta_3 = 0$ for the latter.

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SYMMETRIES AND REDUCTIONS OF THE 2+1-DIMENSIONAL VARIABLE COEFFICIENT BURGERS EQUATION

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We study symmetries of a 2+1-dimensional Burgers equation with variable coefficient. We show that the equation admits an infinite-dimensional Lie algebra as the algebra of its symmetry group which does not have a Virasoro structure whose presence characterize integrability for PDEs in more than 1+1-dimensions. We give a classification of its low-dimensional subalgebras and obtain reduced ODEs. In contrast to an integrable PDE, its reductions to ODEs do not lead to Painlevé type equations. We pick out of them those equations which pass the Painlevé test and obtain their exact solutions.

1 Introduction

The two-dimensional generalized Burgers equation

$$(u_t + uu_x - u_{xx})_x + u_{yy} = 0, \tag{1}$$

which is sometimes referred to as the Zabolotskaya-Khokhlov equation arises in nonlinear acoustics [1,2]. Painlevé analysis of (1) was carried out in [3]. The authors showed that the equation possesses the conditional Painlevé property and obtained its exact solutions by use of truncation. Moreover, the conservation laws and Lie point symmetries of eq. (1) have been investigated in [4], (see also [5] for a comment).

In this paper we shall study the symmetry properties of the variable coefficient version of (1) [6]

$$(u_t + uu_x - u_{xx})_x + s(t)u_{yy} = 0. \tag{2}$$

We first determine the form of the function $s(t)$ which allows a nontrivial symmetry group, i.e. look at a group classification problem, then identify its Lie algebra. We use the two-dimensional subalgebras to perform reductions to ODEs. Finally we analyze the reduced ODEs. Let us mention that a symmetry analysis of a variable coefficient KP equation was performed in [7].

2 The Symmetry Group

The algorithm for determining the form of a symmetry vector field realizing the Lie algebra of the symmetry group for a given system of differential equations is described in any book on the subject (see e.g. ref. [8]). Applying to the equation (2) we obtain the following:

Case I. $s(t)$ arbitrary.

The symmetry algebra is infinite dimensional and a general element is represented by

$$V = X(f) + Y(g), \tag{3}$$

$$X(f) = f(t)\partial_x + f'(t)\partial_u, \tag{4a}$$

$$Y(g) = g(t)\partial_y - \frac{g'(t)}{2s(t)}y\partial_x - \Big(\frac{g'(t)}{2s(t)}\Big)' y\partial_u \tag{4b}$$

where $f(t)$ and $g(t)$ are arbitrary smooth functions and the primes denote time derivatives. The commutation relations are

$$\begin{aligned} &[X(f_1), X(f_2)] = 0, \qquad [X(f), Y(g)] = 0, \\ &[Y(g_1), Y(g_2)] = X\big(\frac{1}{2s}(g_1'g_2 - g_1g_2')\big). \end{aligned} \tag{5}$$

From (5) we see that the symmetry algebra has a Kac-Moody structure but not a Virasoro structure which is typical for 2+1-dimensional integrable equations.

In a more general setting we studied a generalized KP equation with coefficients depending on nine arbitrary functions of one, or two variables [9] from a different point of view. We determined the conditions on coefficients under which the equation is invariant under the Kac-Moody-Virasoro algebra and under the Kac-Moody type algebra only.

Case II. $s(t) = \sigma t^\alpha$, $\sigma =$ const.

In this case, in addition to V in (3) we have the following basis element (a dilation)

$$D_\alpha = x\partial_x + \frac{(3+2\alpha)}{2}y\partial_y + 2t\partial_t - u\partial_u. \tag{6a}$$

Case III. $s(t) = \sigma e^{\alpha t}$, $\sigma =$ const.

In this case the symmetry algebra is represented by V in (3) further extended by the following additional element

$$T_\alpha = \partial_t + \frac{\alpha}{2}y\partial_y. \tag{6b}$$

When $s(t)$ is constant the symmetry algebra is even much larger.
Case IV. s=const.

The symmetry algebra is L_p with two additional generators

$$D_0 = x\partial_x + \frac{3}{2}y\partial_y + 2t\partial_t - u\partial_u, \qquad T_0 = \partial_t. \tag{7}$$

The non-zero commutators amongst D_α, T_α, $X(f)$ and $Y(g)$ are

$$\begin{aligned} &[X(f), D_\alpha] = X(f - 2tf'), \qquad [Y(g), D_\alpha] = Y(\tfrac{(3+2\alpha)}{2}g - 2tg'), \\ &[X(f), T_\alpha] = -X(f'), \qquad\qquad [Y(g), T_\alpha] = Y(\tfrac{\alpha}{2}g - g'). \end{aligned} \tag{8}$$

The Lie algebra L with a basis $X(f)$, $Y(g)$ and D_α or T_α can be written as a semi-direct sum

$$L = \{X(f), Y(g)\} \oplus_s S \tag{9}$$

where $S = \{D_\alpha\}$ or $S = \{T_\alpha\}$. For the last case we write

$$L = \{X(f), Y(g)\} \oplus_s \{D_0, T_0\}. \tag{10}$$

Restricting $f(t)$ and $g(t)$ to be linear polynomials we obtain obvious physical symmetries spanned by

$$\begin{aligned} &X \equiv X(1) = \partial_x, \quad Y \equiv Y(1) = \partial_y, \quad B \equiv X(t) = t\partial_x + \partial_u, \\ &R \equiv Y(t) = -\frac{\sigma}{2}y\partial_x + t\partial_y \end{aligned} \tag{11}$$

which are space translations, Galilei transformations in the x direction and pseudo-rotations, respectively. The six-dimensional subalgebra

$$L_0 = \{T_0, X, Y, D_0, R, B\}$$

is solvable and contains a five-dimensional nilpotent ideal (the Nilradical)

$$NR(L_0) = \{T_0, X, Y, R, B\}$$

3 Reductions to ODEs

Although the original equation is not completely integrable the existence of the infinite dimensional symmetry algebra makes it possible to reduce it to ODEs which are in general do not have the Painleve property, but otherwise we can pick out those equations having the Painleve property among the generically non-Painleve ones. We restrict our attention to the cases II and III

For a systematic reduction we need a classification of low-dimensional subalgebras. Methods available for the classification of finite dimensional algebras to a great degree can be applied to classification of infinite-dimensional ones. Since we need two-dimensional subalgebras to be able to reduce to ODEs, we embed the non-conjugate one-dimensional subalgebras into two-dimensional subalgebras and obtain the following subalgebras:

Abelian:

$$L_{2,1} = \{D_\alpha,\, Y(t^{(3+2\alpha)/4}) + \nu X(t^{1/2})\}, \quad L_{2,2} = \{T_\alpha,\, Y(e^{\alpha t/2}) + \partial_x\},$$

Non-Abelian:

$$L_{2,3} = \{D_\alpha,\, B + Y\left(t^{(5+2\alpha)/4}\right)\}, \quad L_{2,4} = \{T_\alpha,\, Y\left(e^{(\alpha+2)/2t}\right) + X(e^t)\}.$$

3.1 Reduced ODEs and Discussion of Solutions

1.) $L_{2,1}$:

Invariance under the two-dimensional subalgebra $L_{2,1}$ gives the invariant solution

$$u = t^{-1/2}\, H(\rho) + \frac{\nu}{2} y t^{-(5+2\alpha)/4} + \frac{(3+2\alpha)(1+2\alpha)}{64} y^2 t^{-(\alpha+2)},$$

$$\rho = x t^{-1/2} - \nu y t^{-(3+2\alpha)/4} + \frac{3+2\alpha}{16} y^2 t^{-(3+2\alpha)/2}$$

where $H(\rho)$ satisfies the second order ODE

$$H'' - HH' + \left(\frac{\rho}{2} - 1\right) H' + \frac{(1-2\alpha)}{8} H = \frac{(1+2\alpha)(3+2\alpha)}{32} \rho + C \qquad (12)$$

where C is an integration constant. This ODE belongs to the polynomial class of equations of second order [10]. We have not been able to integrate this ODE for any value of the parameter α, nor we have found a first integral of polynomial type. However, for $\alpha = -3/2$ Equation (12) is reduced to the Riccati equation of first order

$$H' - \frac{1}{2} H^2 + \frac{1}{2} \rho H = c_0 \rho + c_1. \qquad (13)$$

By the substitution $H = 2(\hat{H} + \frac{\rho}{4})$, one obtains the normal form of (13)

$$\hat{H}' = \hat{H}^2 + S(\rho), \quad S = -\frac{\rho^2}{16} + \frac{c_0}{2} \rho + \hat{c}_1, \quad \hat{c}_1 = \frac{c_1}{2} - \frac{1}{4}. \qquad (14)$$

This particular value of α is the only value for which (12) passes the Painlevé test which is often an indication for the solution of the equation to be expressed in terms of elementary functions or elliptic functions. Indeed, for a particular

choice of arbitrary constants, namely when $\hat{c}_1 = 0$, and $c_1 = 0$ the general solution of the normalized Riccati equation (14) is expressible in terms of the modified Bessel functions of fractional order in the form

$$h(\rho) = \sqrt{\rho}\left[k_1 I_{1/4}\left(\frac{\rho^2}{32}\right) + k_2 I_{-1/4}\left(\frac{\rho^2}{32}\right)\right],$$

where $\hat{H}$ is obtained from h through logarithmic differentiation as

$$\hat{H} = -\frac{h'}{h}, \quad H = 2(\hat{H} + \frac{\rho}{4}).$$

2.) $L_{2,2}$:
Invariance under $L_{2,2}$ implies that the solution has the form

$$u = H(\rho) + \frac{\alpha^2}{16} y^2 e^{-\alpha t},$$
$$\rho = x - y e^{-\alpha t/2} + \frac{\alpha}{8} y^2 e^{-\alpha t}$$

with $H(\rho)$ satisfying the second order ODE

$$H'' - HH' - H' - \frac{\alpha}{4} H = C + \frac{\alpha^2}{8}\rho.$$

For $\alpha = 0$ it is immediate to see that this equation has a first integral

$$H' - \frac{1}{2}H^2 - H = c_0\rho + c_1 \tag{15}$$

which is a Riccati equation. Again, this equation has an exact solution in terms of Bessel functions.
3.) $L_{2,3}$:
Invariance under $L_{2,3}$ implies that the solution has the form

$$u = t^{-1/2} H(\rho) + y t^{-(5+2\alpha)/4} + \frac{(2\alpha+5)(2\alpha-1)}{32} y^2 t^{-(\alpha+2)},$$
$$\rho = x t^{-1/2} - y t^{-(3+2\alpha)/4} + \frac{2\alpha+5}{16} y^2 t^{-(3+2\alpha)/2}$$

with $H(\rho)$ satisfying a second order ODE of the same form as (12).
4.) $L_{2,4}$:
Invariance under $L_{2,4}$ implies that the solution is

$$u = H(\rho) + y e^{-\alpha t/2} + \frac{(\alpha^2-4)}{16} y^2 e^{-\alpha t},$$
$$\rho = x - y e^{-\alpha t/2} + \frac{(\alpha+2)}{8} y^2 e^{-\alpha t}$$

with $H(\rho)$ satisfying

$$H'' - HH' - \frac{\alpha+2}{4}H - \frac{\alpha^2-4}{8}\rho = C.$$

Once again, for $\alpha = -2$, following an integration we have the Riccati equation

$$H' = \frac{1}{2}H^2 + c_0\rho + c_1$$

as a first integral. Solving this equation provides us with the solutions of the original equation invariant under the two-dimensional subalgebra $\{\partial_t - y\partial_y, \partial_y + e^t(\partial_x + \partial_u)\}$. However, setting $\alpha = 2$ we have

$$H'' - HH' - H = C$$

which is not of the Painlevé type. This is to be expected because transforming from variables $(\rho, H(\rho))$ to $(R, G(R))$ by setting $R = H$, $G = H'$ reduces it to an Abel equation of the second kind

$$GG_R - RG(R) - R = C.$$

For other values of α, unfortunately we failed to solve it completely or to find a first integral.

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A TWO-DIMENSIONAL VERSION OF THE CAMASSA-HOLM EQUATION

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The Camassa-Holm equation of shallow water theory is generalized to fluid flow over a two-dimensional sea bottom. As in the one-dimensional case the resulting equations are of Euler-Poincaré type. The dynamics of the one-dimensional Camassa-Holm equation embeds into the dynamics of the new equations.

1 Introduction

In this note we present the derivation of a higher-dimensional version of the Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0\,. \tag{1}$$

This equation is sometimes also called the Fokas-Fuchssteiner-Camassa-Holm equation, because it was obtained first by Fokas and Fuchssteiner[5], who noted that it is an integrable partial differential equation. The equation was rederived by Camassa and Holm[1] in a fluid dynamical context. In their interpretation, $t \in \mathbf{R}$ is the time variable, $x \in \mathbf{R}$ parametrizes a one-dimensional flat sea bottom and $u(t,x) \in \mathbf{R}$ is the component parallel to this bottom of the spatial velocity field of an ideal, i.e. inviscid and incompressible fluid. Camassa and Holm also discovered that equation (1) has weak soliton solutions of the form

$$u(t,x) = ce^{-|x-ct|}, \quad c \in \mathbf{R}\,, \tag{2}$$

called 'peakons' because their derivative has a discontinuity at $x = ct$.

In this note we generalize the derivation of Camassa and Holm, also presentend in Constantin[4], to fluid flow over a two-dimensional bottom. We note, that in Clarkson *et al.*[3] there is another two-dimensional version of equation (1), which, however, does not seem to have any physical interpretation. Also, the famous Euler-α equations[6] can be regarded as a higher dimensional version of the Camassa-Holm equation, because they have the same geometric structure of being Euler-Poincaré equations (this notion will be explained in the

following section). However, the fluid-dynamical background of both equations is rather different: The Camassa-Holm equation is derived as a shallow-water approximation for ideal fluid flow with a free boundary, whereas the Euler-α equations are an "averaged" version of the Euler equations for an ideal fluid without a free boundary. Since our higher-dimensional Camassa-Holm equations are Euler-Poincaré equations, we start in section 2 by explaining the notion of Euler-Poincaré equations, closely following the presentation of Marsden and Ratiu [9]. In section 3 the two-dimensional Camassa-Holm equations are derived, followed by a discussion of some of their properties in section 4 and some remarks on the numerical treatment of the equations in section 5.

2 Euler-Poincaré Equations

The dynamics of a conservative mechanical system often can be described by Euler-Lagrange equations. If TQ is the tangent bundle of the configuration space Q of the mechanical system with canonical local coordinates $q, \dot{q}$ and if $L : TQ \to \mathbf{R}$ is the Lagrangian of the system , then the Euler-Lagrange equations are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

In case that the configuration space is a Lie group G and the Lagrangian $L : TG \to \mathbf{R}$ is invariant under the lift of the right translation in G to the tangent bundle TG, the dynamics on TG induces a dynamical system on the Lie algebra g which is described by the Euler-Poincaré equations. The precise statement is as follows:

Theorem 1 (Marsden-Ratiu) *Let G be a Lie group with unit element e, TG its tangent bundle, and $g = T_eG$ its Lie algebra with Lie bracket $[\cdot,\cdot]$ and dual space g^*. Let $ad^* : g \times g^* \to g^*$ denote the coadjoint action of g on g^*. Furthermore, let $L : TG \to \mathbf{R}$ be a right-invariant Lagrangian, $l : g \to \mathbf{R}$ its restriction to $g = T_eG$, and let $\frac{\delta l}{\delta \xi}(\xi) \in g^*$ be the derivative of l at $\xi \in g$. To a curve $g(t)$ in G associate the curve $\xi(t) = \dot{g}(t)g(t)^{-1} = TR_{g(t)^{-1}}\dot{g}$ in g. Then the following statements are equivalent:*

1. *$g(t)$ is a solution of the Euler-Lagrange equations for L.*
2. *$\xi(t)$ is a solution of the Euler-Poincaré equations*

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = -ad^*_\xi \frac{\delta l}{\delta \xi}. \tag{3}$$

Note, that the functional L is completely determined by l. (In the reference [9] other equivalent statements involving variational principles are given.)

Euler Poincaré equations are ubiquous in the theory of ideal fluids as the examples below show. We note that our discussion at this point and also in the following sections is on a formal level, i.e. we do not specify function spaces for the occuring maps and vector fields. A basic requirement in all of the following is that it is possible to integrate by parts and that boundary terms vanish upon doing so.

Example 1 (*Euler equations for an ideal fluid*): Let $G = \mathrm{Diff}_{\mathrm{vol}}(D)$ denote the group of volume-preserving diffeomorphisms of a region D in $\mathbf{R}^3$. G is the configuration space of an ideal fluid that completely fills D in the Lagrangian description of motion where one keeps track of the position of each individual fluid particle. The Lie algebra g of G is the algebra of divergence-free vector fields $X_{\mathrm{div}}(D)$ on D. $X_{\mathrm{div}}(D)$ is the Eulerian configuration space of the fluid, i.e. the space of spatial velocity fields. Let $l(u) = \frac{1}{2}\int_D \langle u, u\rangle_3 d^3x$ denote the kinetic energy of an ideal fluid with constant energy $\rho = 1$ (Here, $\langle\cdot,\cdot\rangle_3$ is the standard scalar product in $\mathbf{R}^3$). We identify $X^*_{\mathrm{div}}(D)$ with $X_{\mathrm{div}}(D)$ via the L_2-metric, i.e. $u \in X_{\mathrm{div}}(D)$ is identified with the linear form $\int_D \langle u, \cdot\rangle_3 d^3x$. Then $\frac{\delta l}{\delta u} = u$ and one can show that the Euler-Poincaré equations (3) are equivalent to the Euler equations

$$u_t + (u\cdot\nabla)u = -\nabla p, \quad \mathrm{div} u = 0$$

for an ideal fluid of constant density $\rho = 1$ with pressure function p, moving under the influence of its own inertia in D.

Example 2 (*Camassa-Holm equation*): Let $G = \mathrm{Diff}(\mathbf{R})$ denote the group of diffeomorphisms of $\mathbf{R}$. The Lie algebra g of G is the algebra $X(\mathbf{R})$ of vector fields on $\mathbf{R}$. The Euler-Poincaré equation for the functional $l(u) = \frac{1}{2}\int_{\mathbf{R}} u^2 + u_x^2 dx$ can be shown to be the Camassa-Holm equation (1).

Example 3 (*Euler-α-equations*): Let $G = \mathrm{Diff}_{\mathrm{vol}}(\mathbf{R}^3)$ and $g = X_{\mathrm{div}}(D)$ as in Example 1 and let $l(u) = \frac{1}{2}\int_D \left(\langle u, u\rangle_3 + \alpha \sum_{i=1}^3 \langle u_{x_i}, u_{x_i}\rangle_3\right) d^3x$. Then the corresponding Euler-Poincaré equations are the famous Euler-α equations [6].

3 Derivation of the higher-dimensional Camassa-Holm equations

The equations which will be derived in this section are Euler-Poincaré equations on the Lie algebra $X(\mathbf{R}^2)$ of vector fields on the plane $\mathbf{R}^2$. Here, a vector field $u \in X(\mathbf{R}^2)$ is interpreted as the horizontal component of the spatial velocity field of an ideal fluid. Having fixed the Lie algebra, we still have to supply the functional l on $u \in X(\mathbf{R}^2)$ to be able to write down Euler-Poincaré equations. This functional l is constructed in an approximation process as follows.

We consider a three-dimensional ideal fluid with constant density $\rho = 1$ and a free surface, that moves over a flat bottom. Cartesian coordinates x_1, x_2, y are introduced, such that the flat bottom is at $y = 0$. Putting $x = (x_1, x_2)$, we assume that the position of the free surface is given by $y = \eta(x)$.

Let $(u(x,y), v(x,y)) \in \mathbf{R}^2 \times \mathbf{R}$ denote the value of the (spatial) velocity field at the position $(x,y) \in \{(\bar{x},\bar{y}), \bar{y} \leq \eta(\bar{x})\}$ and let

$$K(u,v) = \frac{1}{2}\int_{\mathbf{R}^2}\int_0^{\eta(x)} \langle u,u\rangle_2 + v^2 \; dyd^2x \tag{4}$$

be its kinetic energy ($\langle\cdot,\cdot\rangle_2$ denotes the canonical scalar product in $\mathbf{R}^2$). As in the original work of Camassa and Holm [1] we make the assumption, that the flow is columnar, i.e. that fluid particles that are on top of each other at one instant of time, are so throughout the motion. In this case, the horizontal velocity component u does not depend on the height variable y: $u = u(x)$. Now we Taylor-expand the vertical velocity component in the y-variable:

$$v(x,y) = v(x,0) + \frac{\partial v}{\partial y}(x,0)y + O(y^2)\,. \tag{5}$$

One has $v(x,0) = 0$ since the fluid at the bottom is assumed to stay at the bottom. We now make the "shallowness"-assumption, that the $O(y^2)$-term in the Taylor expansion can be neglected and that the vertical velocity component has the form $v(x,y) = v_1(x)y$. Therefore the velocity field is given by $(u(x,y), v(x,y)) = (u(x), v_1(x)y)$. This velocity field has to be divergence-free, since the fluid is assumed to be incompressible. Therefore, $v_1(x) = -\mathrm{div}u(x)$, so that we finally arrive at a spatial velocity field of the form $(u(x,y), v(x,y)) = (u(x), -\mathrm{div}u(x)y)$. The corresponding kinetic energy is given by

$$K = \frac{1}{2}\int_{\mathbf{R}^2} \eta(x)\langle u(x), u(x)\rangle_2 + \frac{\eta(x)^3}{3}(\mathrm{div}u(x))^2 d^2x\,. \tag{6}$$

We introduce a small parameter ϵ and expand the free surface $y = \eta(x)$ about the constant height 1:

$$\eta(x) = 1 + O(\epsilon)\,. \tag{7}$$

Then we plug the expansion (7) into the kinetic energy (6). The lowest order term of the resulting expression is the functional

$$\hat{l}(u) = \frac{1}{2}\int_{\mathbf{R}^2} \langle u,u\rangle_2 + \frac{1}{3}(\mathrm{div}u)^2 d^2x\,,$$

which does not depend on η any more. This functional can be embedded in a one-parameter family of real-valued functionals on $X(\mathbf{R}^2)$ by a rescaling process as follows: For $\alpha, \beta > 0$ put $\bar{x} = \alpha x, \bar{\eta}(\bar{x}) = \beta\eta(x), \bar{t} = \gamma t, \bar{u}(\bar{x}) = (\alpha/\gamma)\, u(x)$ and $\delta = \alpha^2/(3\beta^2)$. If one chooses $\gamma = \sqrt{\alpha^3\beta}$, one has

$$K = \frac{1}{2}\int_{\mathbf{R}^2} \bar{\eta}\langle\bar{u},\bar{u}\rangle_2 + \delta^2\bar{\eta}^3\left(\partial_{\bar{x}_1}\bar{u}_1 + \partial_{\bar{x}_2}\bar{u}_2\right)^2 d^2x\,.$$

We write u and η instead of $\bar{u}$ and $\bar{\eta}$, expand as in (7) and keep only the lowest order term to arrive at a family of functionals on the Lie algebra $X(\mathbf{R}^2)$

$$l_\delta(u) = \frac{1}{2}\int_{\mathbf{R}^2} \langle u,u\rangle_2 + \delta(\mathrm{div}\ u)^2 d^2x\,.$$

We put $l = l_1$ and write down the Euler-Poincaré equations corresponding to l. A simple calculation, using integration by parts, shows that $\frac{\delta l}{\delta u} = u - \nabla\mathrm{div}u$. Let $\langle u,v\rangle = \int_{\mathbf{R}^2}\langle u,v\rangle_2 d^2x$ denote the L_2-scalar product of $u, v \in X(\mathbf{R}^2)$. The Euler-Poincaré equations (3) are satisfied if and only if for all $v \in X(\mathbf{R}^2)$

$$\begin{aligned}\langle\frac{d}{dt}\frac{\delta l}{\delta u}, v\rangle &= -\langle \mathrm{ad}^*_u\frac{\delta l}{\delta u}, v\rangle = -\langle\frac{\delta l}{\delta u}, \mathrm{ad}_u v\rangle = \langle\frac{\delta l}{\delta u}, (Dv)u - (Dv)u\rangle \\ &= -\int_{\mathbf{R}^2}\left\langle (Du)^T\frac{\delta l}{\delta u} + \left(D\frac{\delta l}{\delta u}\right)u + \mathrm{div}u\frac{\delta l}{\delta u}, v\right\rangle_2 d^2x\,.\end{aligned}$$

(Here we used the fact that $\mathrm{ad}_u v = -((Dv)u - (Du)v)$, as explained by Marsden and Ratiu [9].) Therefore the Euler-Poincaré equations corresponding to l are given by

$$\frac{d}{dt}(u - \nabla\mathrm{div}u) = -(Du)^T(u - \nabla\mathrm{div}u) - D(u - \nabla\mathrm{div}u)u - \mathrm{div}u(u - \nabla\mathrm{div}u)\,. \quad (8)$$

Putting $m = \frac{\delta l}{\delta u} = u - \nabla\mathrm{div}u$, the equations can be written in the more compact form

$$\frac{d}{dt}m = -(Du)^T m - (Dm)u - (\mathrm{div}u)m\,. \quad (9)$$

Using the functionals l_δ instead of l leads to a one-parameter family of Euler-Poincaré equations

$$\frac{d}{dt}(u - \delta\nabla\mathrm{div}u) = -(Du)^T(u - \delta\nabla\mathrm{div}u) - D(u - \delta\nabla\mathrm{div}u)u - \mathrm{div}u(u - \delta\nabla\mathrm{div}u)\,. \quad (10)$$

We note that the derivation of these equations can be generalized to the case of an "n-dimensional fluid bottom". In particular, for $n = 1$ one recovers the Camassa-Holm equation (1).

4 Some properties of the two-dimensional Camassa-Holm equations

An interesting feature of the equations (8) is, that solutions of the Camassa-Holm equation (1) immediately yield solutions of the equations (8) in the following sense: A simple calculation shows that

Theorem 2 *If $u_1(x)$ is a solution of the Camassa-Holm equation (1), then $u(x_1, x_2) = (u_1(x_1), 0)$ is a solution of the two-dimensional version (8).*

In particular, the peakon solutions (2) of equation (1) yield solutions of equation (8). Furthermore, another simple calculation shows that

Theorem 3 *If $R \in O(2)$ is an orthogonal matrix and if $u(x)$ is a solution of (8), then $Ru(R^T x)$ is a solution of (8) too.*

No existence result is available for equation (8) yet. We note, however, that for the singular case $\delta = 0$, equations (10) constitute a symmetric hyperbolic system of partial differential equations and that the existence results presented in Taylor [10] apply. Analoguous to the one-dimensional situation discussed in in Camassa *et al.* [1,2] or Constantin [4] one can give a Hamiltonian formulation of the two-dimensional Camassa-Holm equations (8), which immediately yields constants of motion for these equations:

On the dual of the semidirect product of the Lie algebra $X(\mathbf{R}^2)$ with the vector space of real valued functions on $\mathbf{R}^2$, one has the following Poisson bracket (compare D.D. Holm *et al.* [7])

$$\{F,H\}(m,\mu) = -\int_{\mathbf{R}^2} \left\langle \mu \frac{\delta F}{\delta m}, \nabla \frac{\delta H}{\delta \mu} \right\rangle_2 - \left\langle \mu \frac{\delta H}{\delta m}, \nabla \frac{\delta F}{\delta \mu} \right\rangle_2 d^2x \qquad (11)$$
$$- \int_{\mathbf{R}^2} \left\langle m, D\left(\frac{\delta H}{\delta m}\right)\left(\frac{\delta F}{\delta m}\right) - D\left(\frac{\delta F}{\delta m}\right)\left(\frac{\delta H}{\delta m}\right) \right\rangle_2 d^2x$$

Let $m = u - \nabla \mathrm{div} u$, as above. For $H = \frac{1}{2}\int_{\mathbf{R}^2} \langle u, u\rangle_2 d^2x$, Hamiltons equations in Poisson bracket form $\dot{F} = \{F, H\}$ for all F are equivalent to

$$\frac{d}{dt} m = -(Du)^T m - (Dm)u - (\mathrm{div} u)m\,, \quad \frac{d}{dt}\mu = -\mathrm{div}(\mu u)\,. \qquad (12)$$

The first equation in (12) is just the two-dimensional Camassa-Holm equation (8), which therefore arises as a projection of the semidirect product system on the m-component. Let $\omega = \mathrm{rot}\left(\frac{m}{\mu}\right)$. Then the functional

$$C_\Phi(M,\mu) = \int_{\mathbf{R}^2} \mu \Phi(\omega/\mu) d^2x$$

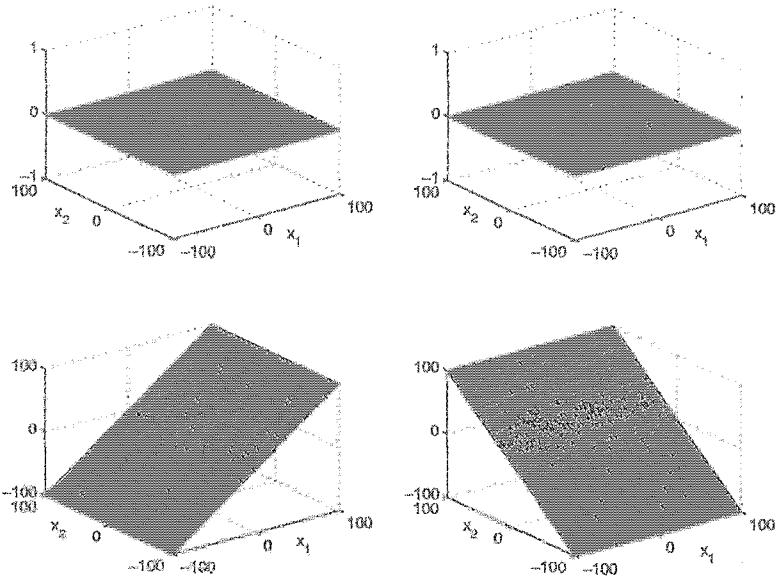

Figure 1: The figure shows numerical results of an multisymplectic algorithm applied to equation (8). More details are given in the main text

is a constant of motion for any system of the form (12), i.e. C_Φ is a Casimir functional for the Poisson bracket $\{\cdot,\cdot\}$. Therefore, the Hamiltonian approach immediately yields a class of constants of motion for the two-dimensional Camassa-Holm equation (8).

5 Numerical treatment of the equations

As noted in section 2, the functional l determines a functional L on the tangent bundle of the group $\mathrm{Diff}(\mathbf{R}^2)$. Furthermore, Theorem 1 describes the relation between solutions $t \mapsto \psi \in \mathrm{Diff}(\mathbf{R}^2)$ of the Euler-Lagrange equations for L and solutions $t \mapsto u = \dot{\psi} \circ \psi^{-1} \in \mathrm{X}(\mathbf{R}^2)$ of the two-dimensional Camassa-Holm equations, i.e. the relation between the Lagrangian and the Eulerian description of fluid motion. An important fact is that solutions of Euler-Lagrange equations can be characterized as critical points of a suitable action functional.

To study the equations (8) numerically we have extended to two space dimensions the multisymplectic-momentum integrator introduced by Kouranbaeva and Shkoller [8] to treat the Camassa-Holm equation. The basic idea

of this method is to pass to the Lagrangian description of the motion and to determine critical points of a discretized version of the corresponding action functional. The pictures show a peakon-type of solution obtained by this method. The lower two pictures show the two components of the fluid particle positions after 70 time-steps of length 0.1. The two components of the initial velocity field $\dot{\psi}|_{t=0}(x) = -e^{23}e^{-(x,x)_2}x$ are shown in the upper two pictures. The corresponding initial positions of the fluid particles are given by the identity map.

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C^∞–SYMMETRIES AND EQUATIONS WITH SYMMETRY ALGEBRA $\mathcal{SL}(2,\mathbb{R})$

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C^∞–Symmetries are used to obtain a step by step method to reduce the order of equations that admit the non-solvable symmetry algebra $sl(2,\mathbb{R})$. A classification of third order equations with symmetry algebra $sl(2,\mathbb{R})$ is given, as well as the corresponding first order reduced equations.

1 Introduction

One of the most utilized methods to calculate exact solutions of differential equations is based on the existence of Lie symmetries (Olver [1], Ovsiannikov [2], Stephani [3]). If an n–th order ordinary differential equation

$$\Delta(x, u^{(n)}) = 0 \tag{1}$$

admits an r–dimensional symmetry Lie algebra, $\mathcal{G}$, then its general solution can be obtained by means of the general solution of an $(n-r)$–th order reduced equation and the solution of an r–th order auxiliary equation. If $\mathcal{G}$ is solvable, then the general solution of the corresponding auxiliary equation can be obtained by r successive quadratures. This fact is due to the possibility of choosing generators $\{X_1, \cdots, X_r\}$ of $\mathcal{G}$ such that the following chains of normal subalgebras hold:

$$< X_1 > \rhd < X_1, X_2 > \rhd \cdots \rhd < X_1, \cdots, X_r >= \mathcal{G}. \tag{2}$$

When the Lie symmetry X_1 is used to reduce the order of Eq. (1), the resulting $(n-1)$–th order equation inherits X_2 as Lie symmetry, and so on. Thus, the order of Eq. (1) can be reduced by means of r one-order successive reductions. In each step, the general solution of the k–th order equation can be obtained from the solution of the $(k-1)$–th order equation by a single quadrature. Nevertheless, if $\mathcal{G}$ is non-solvable, this step by step method of reduction is no longer applicable. The main reason for this is that, in certain stage of the reduction process, at least one of the generators of $\mathcal{G}$ can not be used to proceed with the order reductions. In this case we will say, roughly speaking, that the corresponding symmetries have been lost, or that these generators are lost symmetries, for the reduced equations.

In this paper, we consider $n-$th order differential equations that are invariant under the non-solvable Lie group $SL(2,\mathbb{R})$. A basis $\{X_1, X_2, X_3\}$ of generators of the associated Lie algebra $sl(2,\mathbb{R})$ can be chosen such that the corresponding Lie brackets are given by

$$\begin{aligned}[X_1, X_2] &= X_1, \\ [X_1, X_3] &= 2X_2, \\ [X_2, X_3] &= X_3.\end{aligned} \tag{3}$$

If we use, for instance, the vector field X_1 (resp. X_3) to reduce the order of the equation, the Lie symmetry X_2 is inheritable as Lie symmetry of the reduced equation, but X_3 (resp. X_1) is not inheritable as Lie symmetry. The worst option to reduce the order is to use first X_2, because both, X_1 and X_3, are not inheritable as Lie symmetries of the equation obtained at the first step of the reduction.

In the literature, these lost symmetries are called type I hidden symmetries. Some indirect methods for hidden symmetries have been introduced (Abraham-Shrauner [4]) and some nonlocal symmetries for some specific ordinary differential equations have been studied (Govinder and Leach [5]). Previously, Olver [1] pointed out, in two examples, the usefulness of what he called exponential vector fields (although they are not well-defined vector fields), that can be considered the origin of the hidden symmetries theory.

The importance of the study of hidden symmetries lies in the fact that they can be used to reduce the order of differential equations for which the classical Lie method is not applicable. There exist several examples of equations that, in spite of lacking Lie symmetries, can be reduced or integrated (González-López [6], Olver [7]). Therefore, the classical Lie method must be generalized to explain these processes of reduction or integration. With this aim, a new class of symmetries ($C^\infty-$symmetries), that are useful to reduce the order, has been introduced in Muriel and Romero [8]. The results about this theory, needed to understand what follows in this work, are briefly described in Sec. 2 below.

We prove here that the lost symmetries that appear in reductions of equations with symmetry algebra $sl(2,\mathbb{R})$ can be recovered as $C^\infty-$symmetries for the reduced equations, and, as a consequence, they can be used to reduce successively the order of the equation by three. In particular, the method provides a complete classification of the third order equations that admit the symmetry group $SL(2,\mathbb{R})$. An equivalent general form for these equations had been calculated by Mahomed and Leach [9] by direct methods (see also Ibragimov [10]). Besides, we provide here a simple general form for the first order reduced equations that appear at the last step of the reduction process.

2 $C^\infty-$Symmetries and ordinary differential equations

The $C^\infty-$symmetries of Eq. (1) are well-defined vector fields, in an open subset $M \subset X \times U$ of the space of the variables of the equation, such that Eq. (1) is invariant under some specific way of prolongation. For a function $\lambda \in C^\infty(M^{(1)})$ we define the $n-$th order λ-prolongation of the vector field $X = \xi(x,u)\partial_x + \eta(x,u)\partial_u$, as the following vector field defined on $M^{(n)}$:

$$X^{[\lambda,(n)]} = \sum_{i=0}^{n} (D_x + \lambda)^i \left(\eta(x,u) - \xi(x,u)u_1\right) \partial u_i + \xi(x,u)D_x, \tag{4}$$

where D_x denotes the total derivative operator with respect to x. The $C^\infty(M^{(1)})-$symmetries of Eq. (1) are the vector fields X for which there exist $\lambda \in C^\infty(M^{(1)})$ such that

$$X^{[\lambda,(n)]}(\Delta(x,u^{(n)})) = 0, \quad \text{when} \quad \Delta(x,u^{(n)}) = 0. \tag{5}$$

Sometimes, we also say that X is a $\lambda-$symmetry or a $C^\infty-$symmetry of the equation. This concept includes the concept of Lie symmetry, because Lie symmetries are $\lambda-$symmetries for $\lambda = 0$.

An algorithm that let us reduce the order of the equation can be associated to a given $C^\infty-$symmetry. This method is based on the construction of invariants of $X^{[\lambda,(n)]}$ by derivation of lower order invariants: if $y = y(x,u)$ and $w = w(x,u,u_1)$ are functionally independent invariants of $X^{[\lambda,(n)]}$, then

$$\{y, w, w_1 = \frac{D_x w}{D_x y}, \cdots, w_{n-1} = \frac{D_x w_{n-2}}{D_x y}\} \tag{6}$$

constitutes a complete set of functionally independent invariants of $X^{[\lambda,(n)]}$. By the invariance condition (5), Eq. (1) can be expressed in terms of (6) as an $(n-1)-$th order reduced equation

$$\Delta_r(y, w^{(n-1)}) = 0. \tag{7}$$

If $w = H(y)$ denotes the general solution of the reduced equation Eq. (7), the general solution of Eq. (1) can be obtained by solving the first order equation

$$w(x,u,u_x) = H(y(x,u)). \tag{8}$$

Many of the known order reduction processes, that are not consequence of the existence of Lie symmetries, are a consequence of the invariance of the equation under $C^\infty-$symmetries. Some examples of this fact are the following:

1. Equation (González-López [6]):

$$u_{xx} = u^{-1}u_x^2 + ng(x)u^n u_x + g'(x)u^{n+1}, \quad n \in \mathbb{R} \setminus \{0\} \tag{9}$$

has no Lie symmetries, but admits $X = u\partial_u$ as $\lambda-$symmetry for $\lambda = ng(x)u^n$. In terms of the invariants $\{y = x, w = \frac{u_x}{u} - g(x)u^n\}$ of $X^{[\lambda,(1)]}$, we get the trivial reduced equation $w_y = 0$. The corresponding Eq. (8) is a Bernoulli equation $\frac{u_x}{u} - g(x)u^n = C$, whose general solution provides the solution of Eq. (9).

2. The equation

$$u_{xx} = D_x((x + x^2)e^u), \tag{10}$$

does not have nontrivial Lie symmetries (Olver [7]), but admits $X = \partial_u$ as $\lambda-$symmetry for $\lambda = (x+x^2)e^u$. In terms of $\{y = x, w = u_x - (x+x^2)e^u\}$, that are invariants of $X^{[\lambda,(1)]}$, we get the trivial reduced equation $w_y = 0$. The corresponding Eq. (8) is the trivial reduced equation associated to Eq. (10).

3. The second order equation

$$4u^3u_{xx} + x^2 + 4u^4 + 2u^2 = 0 \tag{11}$$

has no Lie symmetries and can not be solved by standard methods (Muriel and Romero [8]). It admits the $\lambda-$symmetry $X = u\partial_u$ for $\lambda = xu^{-2}$. Both, the reduced equation, $1 + w^2 - w_y = 0$, where $y = x, w = -\left(\frac{u_x}{u} + \frac{x}{2u^2}\right)$, and the auxiliary equation, $2u^2\tan(x + C) + 2uu_x + x = 0$, can be solved by quadrature.

3 $C^\infty-$Symmetries and conservation of symmetries for the Lie algebra $sl(2,\mathbb{R})$.

Let us suppose that Eq. (1) admits the non-solvable Lie algebra $sl(2,\mathbb{R})$ as symmetry algebra. Let $\{X_1, X_2, X_3\}$ be a basis of generators of $sl(2,\mathbb{R})$ such that the corresponding Lie brackets are given by (3). Next we give a procedure to recover the lost symmetries as $C^\infty-$symmetries for the reduced equations. We also show how they can be used to reduce successively the order of the equation Eq. (1) by three when $n > 3$. This procedure is based on some theoretical results, that can be consulted in Muriel and Romero [8]. The study of the reduction process that begin with X_1 and X_3 is similar because the corresponding transformation groups are equivalent under a point transformation.

3.1 *OPTION A: Sequence* $X_1 \rightarrowtail Y_2 \rightarrowtail Z_3$:

1. FIRST STEP: Use of the Lie symmetry X_1:

   Let $\{y, \alpha\}$ be a local system of coordinates such that $X_2 = \partial_\alpha$. By means of the transformation $\{y = y(x,u), w = \alpha_y(x,u,u_1)\}$, we get the reduced equation

   $$\Delta_1(y, w^{(n-1)}) = 0, \quad (y,w) \in M_1. \tag{12}$$

   The vector field $X_2^{(k)}$ is $\pi_{X_1}^{(k)}$−projectable, for $k \in \mathbb{N}$, where $\pi_{X_1}^{(k)}(y, \alpha, \cdots, \alpha_k) = (y, w, \cdots, w_{k-1})$. Then, X_2 is inheritable as Lie symmetry for Eq. (12), and $Y_2 = (\pi_{X_1}^{(1)})_*(X_2^{(1)})$ is the corresponding inherited Lie symmetry of Eq. (12). The vector field $X_3^{(k)}$ is not $\pi_{X_1}^{(k)}$−projectable (hidden symmetry of type I, Abraham-Shrauner [4]).

2. SECOND STEP: Use of the Lie symmetry Y_2 :

   Let $\{z, \beta\}$ be a local system of coordinates such that $Y_2 = \partial_\beta$. By means of the transformation $\{z = z(y,w), \mu = \beta_z(y,w,w_1)\}$, we get the reduced equation

   $$\Delta_2(z, \mu^{(n-2)}) = 0, \quad (z,\beta) \in M_2. \tag{13}$$

   Let π be such that the following diagram is commutative:

   $$\begin{array}{ccc} & \pi & \\ M^{(2)} & \longrightarrow & M_2 \\ \pi_{X_1}^{(2)} \downarrow & & \uparrow \pi_{Y_2}^{(1)} \\ M^{(2)} & \longleftarrow & M_2 \\ & \varphi & \end{array} \tag{14}$$

   where φ stands for the change of variables $(y, w, w_y) \mapsto (z, \beta, \beta_z)$, and $\pi_{Y_2}^{(1)}(z, \beta, \beta_z) = (z, \mu) = (z, \beta_z)$. Let $\widetilde{f}_3 \in C^\infty(M_1)$ be such that $Y_2(\widetilde{f}_3) = -\widetilde{f}_3$ and let us denote $f_3 = (\pi_{Y_2}^{(1)})^*(\widetilde{f}_3)$. The function f_3 is, by construction, an $X_1^{(1)}$−invariant function. The vector field $f_3 X_3^{(2)}$ is π−projectable. The projection $Z_3 = (\pi)_*(f_3 X_3^{(2)})$ is a C^∞−symmetry of equation Eq. (13), for some function λ_3.

3. THIRD STEP: Use of the C^∞−symmetry Z_3:

   Let $\{s, \gamma\}$ be the canonical coordinates for Z_3. We calculate a first order invariant ρ of $Z_3^{[\lambda_3,(1)]}$. By means of the transformation $\{s = s(z,\mu), \rho = \rho(z, \mu, \mu_z)\}$, we get a new order reduction:

   $$\Delta_3(s, \rho^{(n-3)}) = 0. \tag{15}$$

3.2 OPTION B: Sequence $X_2 \rightarrowtail Y_1 \rightarrowtail Z_3$:

1. FIRST STEP: Use of the Lie symmetry X_2:

   Let $\{y, \alpha\}$ be the canonical coordinates for X_2. By means of the transformation $\{y = y(x,u), w = \alpha_y(x,u,u_1)\}$, we get the reduced equation

   $$\Delta_1(y, w^{(n-1)}) = 0, \quad (y,w) \in M_1. \tag{16}$$

   The vector fields $X_1^{(k)}, X_3^{(k)}$ are not $\pi_{X_2}^{(k)}$–projectable (hidden symmetries of type I). Let $f_1, f_3 \in C^\infty(M)$ be two functions such that

   $$X_2(f_1) = f_1, \quad X_2(f_3) = -f_3. \tag{17}$$

   The vector fields $f_1 X_1^{(1)}, f_3 X_3^{(1)}$ are $\pi_{X_2}^{(1)}$–projectable, and the projections

   $$Y_1 = (\pi_{X_2}^{(1)})_*(f_1 X_1^{(1)}), \quad Y_3 = (\pi_{X_2}^{(1)})_*(f_3 X_3^{(1)}) \tag{18}$$

   are C^∞–symmetries of Eq. (16) for some functions λ_1 and λ_3.

2. SECOND STEP: Use of the C^∞–symmetry Y_1 (the use of Y_3 leads to similar results):

   Let $\{z, \beta\}$ be canonical coordinates for Y_1. Let μ be an invariant of $Y_1^{[\lambda_1,(1)]}$. By means of the transformation $\{z = z(y,w), \mu = \mu(y,w,w_1)\}$, the order of Eq. (16) can be reduced:

   $$\Delta_2(z, \mu^{(n-3)}) = 0, \quad (z,\mu) \in M_2. \tag{19}$$

   It can be checked that

   $$[Y_1^{[\lambda_1,(k)]}, Y_3^{[\lambda_3,(k)]}] = c_1 Y_1^{[\lambda_1,(k)]} + c_3 Y_3^{[\lambda_3,(k)]}, \tag{20}$$

   for some functions $c_1, c_3 \in C^\infty(M_1)$. Let $g_3 \in C^\infty(M_1)$ be a function such that $Y_1(g_3) = -c_3 g_3$. Then $g_3 Y_3$ is $\pi_{Y_1}^{[\lambda_1,(1)]}$–projectable, where $\pi_{Y_1}^{[\lambda_1,(1)]}(z,\beta,\mu) = (z,\mu)$. The projection $(\pi_{Y_1}^{[\lambda_1,(1)]})_*(g_3 Y_3)$ is a C^∞–symmetry of the equation Eq. (19), for some function $\widetilde{\lambda}_3$.

3. THIRD STEP: Use of Z_3 :

   We introduce canonical coordinates $\{s, \gamma\}$ for Z_3 and let ρ be a first order invariant of $Z_3^{[\widetilde{\lambda}_3,(1)]}$. In terms of $\{s, \rho\}$, Eq. (19) can be written as an $(n-3)$–th order equation

   $$\Delta_3(s, \rho^{(n-3)}) = 0. \tag{21}$$

4 General method to reduce equations admitting $sl(2,\mathbb{R})$.

There are four different actions of the group $SL(2,\mathbb{R})$ on a two dimensional real manifold (González-López et al. [11]). Each one of these actions can be modeled by the transformation group generated by the following vector fields:

$$\text{Case 1}: X_1 = \partial_x, \quad X_2 = x\partial_x, \quad X_3 = x^2\partial_x. \tag{22}$$

$$\text{Case 2}: X_1 = \partial_x + \partial_u, \quad X_2 = x\partial_x + u\partial_u, \quad X_3 = x^2\partial_x + u^2\partial_u. \tag{23}$$

$$\text{Case 3}: X_1 = \partial_x, \quad X_2 = x\partial_x + u\partial_u, \quad X_3 = x^2\partial_x + 2xu\partial_u. \tag{24}$$

$$\text{Case 4}: X_1 = \partial_x, \quad X_2 = x\partial_x + u\partial_u, \quad X_3 = (x^2 - u^2)\partial_x + 2xu\partial_u. \tag{25}$$

By means of a change of variables in Eq. (1), we can assume that the symmetry algebra is generated by one of the vector fields $\{X_1, X_2, X_3\}$ given in (22)-(25). For each of these four cases we have studied the different reduction processes described in the previous section. We work out in more detail the case 1. Since option A is the traditional method of reduction (except that Z_3 is not a Lie symmetry), we study option B of case 1, to show how the method works in practice.

4.1 Case 1-Option B: Sequence $X_2 \rightarrowtail Y_1 \rightarrowtail Z_3$:

1. FIRST STEP: Use of the Lie symmetry X_2:

   The vector field $X_2 = x\partial_x$ can be written in the canonical form $X_2 = \partial_\alpha$ by using coordinates $y = u, \alpha = \ln(x)$. By means of the transformation $\{y = u, w = \frac{1}{xu_x}\}$, we get the corresponding Eq. (16). The vector fields

   $$X_1^{(1)} = e^{-\alpha}\partial_\alpha - e^{-\alpha}\alpha_y\partial_{\alpha_y}, \quad X_3^{(1)} = e^{\alpha}\partial_\alpha + e^{\alpha}\alpha_y\partial_{\alpha_y}, \tag{26}$$

   are not $\pi^{(1)}_{X_2}$–projectable (hidden symmetries of type I). Two functions satisfying Eqs. (17) are given, in variables $\{y, \alpha\}$, by $f_1 = e^{\alpha}$ and $f_3 = e^{-\alpha}$. The inherited C^∞–symmetries in (18) are given by $Y_1 = -w\partial_w$, for $\lambda_1 = -w$, and $Y_3 = w\partial_w$, for $\lambda_3 = w$.

2. SECOND STEP: Use of the C^∞–symmetry Y_1:

   Since $Y_1^{[\lambda_1,(1)]} = -w\partial_w + (w^2 - w_y)\partial_{w_y}$, the canonical coordinates for Y_1 are $\{z = y, \beta = -\ln(w)\}$ and $\mu = \beta_z - e^{-\beta}$ is a first order invariant. By means of the transformation $\{z = y, \mu = \frac{-w_y}{w} - w\}$, we get the corresponding Eq. (19). It can be checked that c_1 and c_3 in (20) are given by $c_1 = c_3 = -1$. Thus, a corresponding function g_3 is given by $g_3 = e^{\beta}$.

Table 1. Classification of third order equations with symmetry algebra $sl(2,\mathbb{R})$.

Original equations	Reduced equations
$u_3 = \frac{3u_2^2}{2u_1} - 2u_1^3 C(u)$	$\gamma_s = \gamma^2 + C(s)$
$u_3 = \frac{3u_2^2}{2u_1} + \frac{u_1^2}{2(u-x)^2 C\left((2u_1+2u_1^2+u_2(-u+x))u_1^{-3/2}\right)}$	$\gamma_s = (2+2\gamma^2-\gamma s)C(s)$
$u_3 = \frac{-1}{8u^2 C(u_1^2-2uu_2)}$	$\gamma_s = (4\gamma^2 - s)C(s)$
$u_3 = \frac{3u_1u_2^2}{1+u_1^2} + \frac{(1+u_1^2)^2}{2u^2 C\left((1+u_1^2+uu_2)(1+u_1^2)^{-3/2}\right)}$	$\gamma_s = (s+\sin(2\gamma))C(s)$

The vector field $g_3 Y_3^{[\lambda_3,(1)]}$ is $\pi_{Y_1}^{[\lambda_1,(1)]}$−projectable and its projection, in variables $\{z,\mu\}$, is $Z_3 = -2\partial_\mu$, that is a C^∞−symmetry for $\lambda_3' = -\mu$.

3. THIRD STEP: Use of Z_3 :

   In variables $\{z,\mu,\mu_z\}$ the first prolongation is given by: $Z_3^{[\lambda_3',(1)]} = -2\partial_\mu + 2\mu\partial_{\mu_z}$. We introduce canonical coordinates $\{s = z, \gamma = -\frac{1}{2}\mu\}$ for Z_3. Clearly, $\rho = \gamma_s - \gamma^2 = -\frac{1}{2}\mu_z - \frac{1}{4}\mu^2$ is an invariant of $Z_3^{[\lambda_3,(1)]}$. In terms of $\{s,\rho\}$, Eq. (19) can be written as an $(n-3)$−th order equation of the form

$$\Delta(s,\rho^{(n-3)}) = 0. \tag{27}$$

5 Classification of equations

When $n = 3$, instead of Eq. (27), we obtain an equation of the form $\rho = C(s)$, i.e., we have obtained the general form of the first order equations that appears after two order reductions of a third order equation that admits the symmetry algebra $sl(2,\mathbb{R})$ of type (22):

$$\gamma_s - \gamma^2 = C(s). \tag{28}$$

As a consequence of the step by step method of reduction we have described, a complete classification of the third order ordinary differential equations that admit the non-solvable Lie algebra $sl(2,\mathbb{R})$ as symmetry algebra can be carried out. If we write each of the first order reduced equations corresponding to cases (22)-(25), in terms of the original system of coordinates, we obtain the third order differential equations that admit $sl(2,\mathbb{R})$ as symmetry algebra. They are shown in Table 1, as well as the associated reduced equations.

6 Conclusions

When the classical Lie method is used to reduce the order of any ordinary differential equation admitting the three-dimensional non-solvable Lie algebra $sl(2,\mathbb{R})$ as symmetry algebra, then at least one of its generators is lost in the reduction process. In this paper we have proved that the method of reduction by using $C^\infty-$symmetries can be applied to carry out three successive one-order reductions, if the order of the original equation is $n > 3$. If $n = 3$, after two order reductions, a first order differential equation is obtained. At this last step of the reduction, the Lie symmetry that has been lost can be recovered as a $C^\infty-$symmetry. This fact allows us to give a complete classification of the third order equations that admit $sl(2,\mathbb{R})$, as well as the corresponding reduced equations (first order equations).

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GENERALIZATIONS OF GORDON'S THEOREM

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Gordon's theorem claims that given a Hamiltonian system all of whose solutions are periodic, the period of a solution depend only on the value of the Hamiltonian function on the trajectory of this solution. Generalizations are obtained for the case of invariant isotropic tori of arbitrary dimension k (rather than $k = 1$), which fiber either all the phase space or some submanifold of this. One supposes that the system has some collection of k first integrals in involution, such that the corresponding vector fields are tangent to these tori. Then the frequencies of quasiperiodic motion on such a torus depend only on the values of these first integrals on the torus. This is true also for the circular action functions, but sufficient conditions are essentially different in these two cases.

Introduction

In this note we consider two generalizations of Gordon's theorem for systems defined on the phase space M and which are integrable on a submanifold $N \subseteq M$.

A short foirmulation of the main statement is the following. Let the system defined by the Hamiltonian H on M be "torus Hamiltonian integrable" on the submanifold $N \subseteq M$; let $z := (z_1, ..., z_k)$ be a collection of integrals of motion which are independent and pairwise in involution on N and define there the integrability of the system. Then the frequencies ω of the quasiperiodic motions on the k-dimensional tori $\Lambda \simeq \mathbf{T}^k$ fibering N depend only on z:

$$\omega = \omega(z) \quad , \quad z = z(\Lambda) \quad , \quad \omega = (\omega_1, ..., \omega_k) \; .$$

The same statement also holds, under essentially weaker conditions, for the "circular action functions" I: i.e. we have

$$I = I(z) \quad , \quad z = z(\Lambda) \quad , \quad I = (I_1, ..., I_k) \; .$$

1 Background

First of all it is necessary to understand what integrability on submanifold is.

The object under study is an autonomous, generally non integrable, Hamiltonian system. Such a system is defined by its Hamiltonian function $H : M \to \mathbf{R}$, where M is $2n$-dimensional and is a symplectic manifold with symplectic structure ω^2.

Integrable systems are relatively simply contructed and sufficiently well explored. But, what is the structure of phase flow for nonintegrable systems ? It is well known that many of such systems are integrable only on some invariant submanifold of the complete phase space M.

We will be mainly interested in the torus integrability on a submanifold $N \subseteq M$. This means that N is locally trivially fibered in k-dimensional invariant tori

$$\Lambda^b : \cup_{b \in B} \Lambda^b = N \ , \ \Lambda^b \simeq \mathbf{T}^k \ ,$$

where B is a manifold of parameters b.

The invariance means that $g_H^t \Lambda^b = \Lambda^b$ for all $t \in \mathbf{R}$, where $\{g_H^t, t \in \mathbf{R}\}$ is the phase flow of the system.

We suppose that the motion on these tori is quasi-periodic.

The study of the integrability submanifolds is very important. First of all these submanifolds are very interesting for applications; second, in the neighbourhood of these manifolds the structure of phase flow is most simple. Apparently, the investigation of the integrability submanifold is the simplest and most relevant step in the study of nonintegrable systems.

2 Gordon's theorem.

As mentioned above, we will provide two generalizations of Gordon's theorem. Let us first of all recall the original formulation [1] of this.

Theorem (Gordon). *Let all the solutions of the Hamiltonian system (M^{2n}, ω^2, H) be periodic and let their trajectories locally trivially fiber the phase space M^{2n}. Then the period T and hence the frequence $\omega = 2\pi/T$ are functions only of the Hamiltonian H, i.e. $T = T(H)$, $\omega = \omega(H)$. The circular action function I also depend only on H: i.e. $I = I(H)$.*

The function I is defined as the integral

$$I = I(m) = \frac{1}{2\pi} \int_{\gamma_m} \alpha \ .$$

Here $m \in M$, γ_m is the trajectory of the system such that $m \in \gamma_m$, and α is a differential one-form such that $d\alpha = \omega^2$.

If such a α does not exist then I can be defined as

$$I(m) = \frac{1}{2\pi} \int \int_{\Pi_m} \omega^2$$

where $\partial \Pi_m = \gamma_m - \gamma^0$, with γ^0 some fixed trajectory.

It is then easy to show that $H = H(I)$ and $\omega = \omega(I) = dH(I)/dI$.

An example, classical and simple, is provided by the two-bodies problem when the two bodies attract each other with a force $f = -\kappa/r^2$. This system has three degrees of freedom; all trajectories with negative energy $H < 0$ are closed, besides exceptional (collision) ones.

A five parameteric family of such trajectories fibers the space M^6 and four-parametric families fiber the level surfaces $H^{-1}(h)$ of the function H for $h < 0$. On each such surface, the period T of any trajectory is the same.

At first sight, this situation is very strange and surprising. Really, the (three-parametric) group $SO(3)$ acts on M^6 and preserves the symplectic structure ω^2 and the Hamiltonian function H. However, the number of parameters of closed curves on $H^{-1}(h)$ is equal to four. Obviously, the phase flow of the system also acts, but it cannot connect different trajectories, as its orbits are just these trajectories.

The explanation is that the system has a fifth integral beyond H and the three component M_1, M_2, M_3 of the angular momentum $\mathbf{M}$ corresponding to the action of $SO(3)$. This fifth integral provides closedness of trajectories. The phase flow of this integral together with phase flows of M_1, M_2, M_3 distribute an orbit on the complete surface $H^{-1}(h)$, $h < 0$.

These phase flows are symplectomorphisms and preserve H. Therefore they also preserve the period T. This is a good explanation, but only in simple cases.

3 Generalizations of Gordon's theorem

The generalizations of Gordon's theorem we consider here will be in two directions. Firstly, we shall consider the fibering on k-dimensional tori. Secondly, we shall consider the fibering of a submanifold $N \subseteq M$. The precise formulations are as follows.

Let (M^{2n}, ω^2) be a symplectic manifold. Let $Z = (Z_1, ..., Z_k)$ be some collection of functions $Z_i : M \to \mathbf{R}$; denote by JdZ_i the Hamiltonian vector

field defined by the function Z_i $(i = 1, ..., k)$, and by $JdZ_i(m)$ its value in the point $m \in M$. Let $N \subseteq M$ be some submanifold of M. Assume Z, N satisfy the following conditions.

(a) For all $m \in N$ and all $i = 1, ..., k$, $JdZ_i(m) \in \mathrm{T}_m N$, with $\mathrm{T}_m N$ the tangent space to the submanifold N in m.

(b) For all $i, j = 1, ..., k$, $\{Z_i, Z_j\}|_N = 0$, where $\{.,.\}$ is the Poisson bracket and $|_N$ denotes its restriction to N.

(c) Let Θ be a field of hyperplanes on N given by linear spans of the vectors JdZ_i, $i = 1, ..., k$. Then this field Θ is integrable in the sense of Frobenius. The integral surfaces Λ of Θ are compact k-dimensional submanifolds of N. These surfaces locally trivially fiber N.

Theorem A. *In this case (i.e. if conditions (a),(b),(c) are satisfied) each surface Λ is a k-dimensional torus; moreover the Z are constant on each Λ. The circular action functions $I = (I_1, ..., I_k)$ are locally constant on each common level surface $\widetilde{Z}^{-1}(z) \cap N$: that is, $I|_K$, where K is any connected component of the set $Z^{-1}(z) \cap N = \mathrm{const}$.*

Let the following addtional conditions be also satisfied:

(d) The map $\widetilde{Z} : N \to \mathbf{R}^k$, defined as $\widetilde{Z} = Z|_N$, be regular, i.e. let $\widetilde{Z}$ have no critical point where $\widetilde{Z} = (\widetilde{Z}_1, ..., \widetilde{Z}_k)$, $\widetilde{Z}_i = Z_i|_N$.

(e) There exists a system on M with Hamiltonian function H such that $dH = \sum_{i=1}^{k} \lambda_i dZ_i$, where $\lambda_i = \lambda_i(m)$, $m \in M$.

Theorem B. *Under conditions (a)-(e), the restriction $\widetilde{H} := H|_N$ is locally a function only of $\widetilde{Z} = Z|_N$. Moreover locally*

$$I = I(\widetilde{Z}) \ , \ \widetilde{Z} = \widetilde{Z}(I) \ , \ \omega = \omega(I) = \frac{\partial \widetilde{H}(I)}{\partial I} \ .$$

In particular the frequencies ω of quasi periodical motions on tori Λ depend only on Z, i.e. $\omega = \omega(\widetilde{Z})$.

Remark 1. In case of theorem B, we have $\dim N \geq 2k$. If $\dim N = 2k$, then N is a symplectic submanifold of M.

Remark 2. If $N \subseteq M$ i s a symplectic submanifold, then condition (d), i.e. $\mathrm{rank}\widetilde{Z}_* = k$, automatically follows from conditions (a), (b), (c).

Remark 3. In case of a symplectic submanifold N, the theorem B is a sufficiently simple consequence of the main theorem of the paper [2].

Remark 4. In the case of a non-symplectic submanifold $N \subset M$, conditions (a), (b), (c) and (e) are not sufficient for the validity of theorem B, as shown by the following example.

Example. Let us consider a 4-dimensional symplectic manifold $M^4 = \mathbf{R}^2_J \times \mathbf{T}^2_\psi$ where J, ψ are canonical action-angle "coordinates", ψ being defined modulo 2π. Consider on this the integrable system with Hamiltonian $H = J_1 J_2$. Let us take as N the three-dimensional submanifold

$$N \equiv N^3 := \{(J,\psi) \in M^4 \mid J_1 > 0\,,\ J_2 = 0\}\,.$$

Then the frequencies Ω_1, Ω_2 of quasi periodic motions on tori on $\{J = \text{const}\} \subset N^3$ are given by

$$\Omega_1 = \frac{\partial H}{\partial J_1} = J_2 = 0 \quad , \quad \Omega_2 = \frac{\partial H}{\partial J_2} = J_1 \neq 0 \quad .$$

The connected submanifold $N^3 \subset M^4$ is trivially fibered by the closed trajectories γ of the system. It lies on an isoenergetic surface, $N^3 \subset H^{-1}(0)$. The frequency ω of periodic solutions with these trajectories $\gamma \subset N$ is not constant on N^3, as $\omega = J_1 \neq 0$.

This contradicts the statement that the frequency ω must be a function of H alone, and hence constant on manifolds $N \subset H^{-1}(0)$. Indeed, condition (d) is not satisfied in this example.

4 About proofs.

The proofs of these theorems use in particular the following simple facts.

Consider first of all the definition of the Hamiltonian vector field $X = JdH$. This is defined by

$$dH(\xi) \;=\; \omega^2(\xi, X(m)) \quad \forall \xi \in \mathrm{T}_m M\,.$$

Using this formula one obtains for functions A and B defined on M:

$$\{A,B\} \;:=\; dA\,(JdB) \;=\; \omega^2(JdB, J\mathrm{d}A) \;-\; dB\,(JdA)\,.$$

We also use the formula

$$[JdA\,,\ JdB] \;=\; Jd\,\{A,B\}$$

connecting the Lie bracket and the Poisson bracket.

Definitions. The operator $J_m : \mathrm{T}^*_m M \to \mathrm{T}_m M$ is called the *Hamilton operator*. Let $\widetilde{\omega}^2$ be the restriction of ω^2 to $N \subseteq M$, $\widetilde{\omega}^2 := \omega^2|_N$. The submanifold $N \subset M^{2n}$ is called *isotropic* if $\widetilde{\omega}^2 = 0$, and *lagrangian* if $\dim N$ is maximal among isotropic submanifolds, i.e. if $\dim N = n$. The submanifold $N \subseteq M$ is called *symplectic* if $\widetilde{\omega}^2$ is nondegenerate, i.e. if $(N, \widetilde{\omega}^2)$ is a symplectic manifold.

Using these, it is easy to prove the following

Lemma. *Let the system with Hamiltonian $H : M \to \mathbf{R}$ be defined on the symplectic manifold (M^{2n}, ω^2). Let $N \subseteq M$ be a symplectic submanifold of M, invariant under the local phase flow of the system,*

$$J_m\, dH \in \mathrm{T}_m N \quad \forall m \in N\ .$$

Denote by $\widetilde{J}$ the Hamilton operator corresponding to the symplectic manifold $(N, \widetilde{\omega}^2)$, and let $\widetilde{H} = H|_N$. Then:

(a) The restriction to N of the Hamiltonian vector field JdH coincides with the Hamiltonian vector field $\widetilde{J}d\widetilde{H}$, $(JdH)_N = \widetilde{J}d\widetilde{H}$.

(b) Let A be another function defined on M. Then, with $\{.,.\}_N$ the Poisson bracket on the symplectic manifold $(N, \widetilde{\omega}^2)$ and $\widetilde{A} = A|_N$, we have $\{A, H\}|_N = \{\widetilde{A}, \widetilde{H}\}_N$.

These simple facts allow to reduce the theorem B in the particular case of a symplectic submanifold $N \subset M$ to the main theorem of my article [2] and its corollary, see section 1. The general case can be reduced to this case of a symplectic submanifold $N \subset M$.

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MOVING FRAMES: A BRIEF SURVEY

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The aim of this note is to survey the recent literature on the new equivariant theory of moving frames developed by the author and Mark Fels[14,15]. The classical Cartan theory[11,18], as well as its more rigorous later revival[17,22], has a fairly limited range of geometrical applications. In contrast, the new equivariant theory can be systematically applied to completely general transformation groups, including infinite-dimensional Lie pseudo-groups. The full range of new applications is surprisingly broad, including complete classification of differential invariants and their syzygies, general equivalence and symmetry problems based on differential invariant and joint invariant signatures, classical invariant theory and algebra, computer vision and object recognition, the calculus of variations, Poisson geometry and solitons, and symmetry-based numerical approximation theory.

This note begins with a very brief outline of the key construction in the finite-dimensional Lie group context, illustrated by a very simple, classical example. The second part of the note lists all current references for the various applications. There are several more detailed surveys available[16,36,37,38,41]. A very elementary introduction can be found in Chapter 8 of my recent book[35]. The full details of the method can be found in the original paper with Fels[15]. Further important developments of the general construction can be found in the recent paper with Kogan[26]. All of my papers are available on my web site.

The Basic Construction: Let G be an r-dimensional Lie group acting smoothly on an m-dimensional manifold M. The crucial idea is to decouple the moving frame theory from reliance on any form of frame bundle. In other words, in general *Moving frames* $\neq$ *Frames*! A careful study of Cartan's analysis of projective curves[11], reveals that he was well aware of this distinction, that, unfortunately, was not properly appreciated by most subsequent developers of the method.

Definition 1 A *moving frame* is a smooth, G-equivariant map $\rho : M \to G$.

The group G acts on itself by left or right multiplication. If $\rho(z)$ is any right-equivariant moving frame then $\widetilde{\rho}(z) = \rho(z)^{-1}$ is left-equivariant and conversely. In geometrical situations, one can identify left-equivariant moving

frames with the geometrical frame-based versions, but these identifications break down when dealing with more general group acctions.

Theorem 2 *A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.*

Recall that G acts *freely* if the group element that fixes a point of M is the identity, i.e., $g \cdot z = z$ for some $z \in M$ if and only if $g = e$. This implies that the orbits all have the same dimension as G itself. *Regularity* requires that, in addition, each point $x \in M$ has a system of arbitrarily small neighborhoods whose intersection with each orbit is connected.

Of course, most interesting group actions are *not* free, and therefore do not admit moving frames in the sense of Definition 1. There are three basic methods for converting a non-free action into a free action. The first is to look at the product action of G on several copies of M, leading to joint invariants, also known as "semi-differential invariants" in the computer vision literature[12,32]. The second is to prolong the group action to jet space, which is the natural setting for the traditional moving frame theory, and leads to differential invariants. Combining the two methods of prolongation and product will lead to joint differential invariants. In applications of symmetry constructions to numerical approximations of derivatives and differential invariants, one requires a unification of these different actions into a new common framework, called multispace[40].

The practical construction of a moving frame is based on Cartan's method of *normalization*[11,23].

Theorem 3 *Let G act freely and regularly on M, and let $K \subset M$ be a (local) cross-section to the group orbits. Given $z \in M$, let $g = \rho(z)$ be the unique group element that maps z to the cross-section: $g \cdot z = \rho(z) \cdot z \in K$. Then $\rho : M \to G$ is a right moving frame.*

Given local coordinates $z = (z_1, \ldots, z_m)$ on M, let $w(g, z) = g \cdot z$ be the explicit formulae for the group transformations. The right moving frame $g = \rho(z)$ associated with a *coordinate cross-section* $K = \{ z_1 = c_1, \ldots, z_r = c_r \}$ is obtained by solving the *normalization equations*

$$w_1(g, z) = c_1, \qquad \ldots \qquad w_r(g, z) = c_r, \tag{1}$$

for the group parameters $g = (g_1, \ldots, g_r)$ in terms of the coordinates $z = (z_1, \ldots, z_m)$. Substituting the moving frame formulae into the remaining transformation rules leads to a complete system of invariants for the group action. These are, in fact, the local cross-section coordinates of the cross-section representative or *normal form* $k = \rho(z) \cdot z \in K$ of $z \in M$.

Theorem 4 *If $g = \rho(z)$ is the moving frame solution to the normalization equations* (1)*, then the functions*

$$I_1(z) = w_{r+1}(\rho(z), z), \qquad \dots \qquad I_{m-r}(z) = w_m(\rho(z), z), \tag{2}$$

form a complete system of functionally independent invariants.

Example 5 Let us illustrate the theory with a very simple, well-known example: curves in the Euclidean plane. The orientation-preserving Euclidean group SE(2) acts on $M = \mathbb{R}^2$, mapping a point $z = (x, u)$ to

$$y = x\cos\theta - u\sin\theta + a, \qquad v = x\sin\theta + u\cos\theta + b. \tag{3}$$

the action is not free, and so to construct a moving frame we prolong to the jet space. (Alternatively, one could "prolong" by taking Cartesian products.) For a parametrized curve $z(t) = (x(t), u(t))$, the prolonged group transformations

$$v_y = \frac{dv}{dy} = \frac{x_t\sin\theta + u_t\cos\theta}{x_t\cos\theta - u_t\sin\theta}, \qquad v_{yy} = \frac{d^2v}{dy^2} = \frac{x_tu_{tt} - x_{tt}u_t}{(x_t\cos\theta - u_t\sin\theta)^3}, \tag{4}$$

and so on, are found by successively applying implicit differentiation operator

$$D_y = \frac{1}{x_t\cos\theta - u_t\sin\theta}\, D_t \tag{5}$$

to v. The classical Euclidean moving frame for planar curves[18], follows from the cross-section normalizations

$$y = 0, \qquad v = 0, \qquad v_y = 0. \tag{6}$$

Solving for the group parameters $g = (\theta, a, b)$ leads to the right-equivariant moving frame

$$\theta = -\tan^{-1}\frac{u_t}{x_t}, \qquad a = -\frac{xx_t + uu_t}{\sqrt{x_t^2 + u_t^2}}, \qquad b = \frac{xu_t - ux_t}{\sqrt{x_t^2 + u_t^2}}. \tag{7}$$

The inverse group transformation $g^{-1} = (\widetilde{\theta}, \widetilde{a}, \widetilde{b})$ is the classical left moving frame[11,18]: one identifies the translation component $(\widetilde{a}, \widetilde{b}) = (x, u) = z$ as the point on the curve, while the columns of the rotation matrix $R_{\widetilde{\theta}} = (\mathbf{t}, \mathbf{n})$ are the unit tangent and unit normal vectors. Substituting the moving frame normalizations (7) into the prolonged transformation formulae (4), results in the fundamental differential invariants

$$v_{yy} \longmapsto \kappa = \frac{x_tu_{tt} - x_{tt}u_t}{(x_t^2 + u_t^2)^{3/2}}, \qquad v_{yyy} \longmapsto \frac{d\kappa}{ds}, \qquad v_{yyyy} \longmapsto \frac{d^2\kappa}{ds^2} + 3\kappa^3, \tag{8}$$

where $D_s = (x_t^2 + u_t^2)^{-1/2}\, D_t$ is the arc length derivative — which is itself found by substituting the moving frame formulae (7) into the implicit differentiation operator (5). A complete system of differential invariants for

the planar Euclidean group is provided by the curvature and its successive derivatives with respect to arc length: $\kappa, \kappa_s, \kappa_{ss}, \ldots$.

The one caveat is that the first prolongation of SE(2) is only locally free on J^1 since a 180° rotation has trivial first prolongation. The even derivatives of κ with respect to s change sign under a 180° rotation, and so only their absolute values are fully invariant. The ambiguity can be removed by including the second order constraint $v_{yy} > 0$ in the derivation of the moving frame. Extending the analysis to the full Euclidean group E(2) adds in a second sign ambiguity which can only be resolved at third order[39].

We now survey of the current applications of this basic construction.

Classification of Differential Invariants and Syzygies: The moving frame method was used to completely solve the main classification problems for differential invariants[15]. The recurrence formulae relating the differentiated invariants and the normalized invariants, as in (8), are constructed by purely infinitesimal methods, using only linear algebra and differentiation. The recurrence formulae lead to a complete solution to the problem of classifying syzygies (functional relations) among differential invariants. The moving frame construction was used to clarify the singularities and geometric structure of prolonged group actions on submanifolds[37]. These ideas were extended[26,27] to construct a group-invariant version of the full variational bicomplex[1,2,42].

Inductive Construction: Kogan[24,25] establishes a useful inductive method for building a moving frame for a large group based on a moving frame for a subgroup. The inductive algorithm leads to the general formulae relating the differential invariants of groups and their subgroups.

Joint Invariants and Joint Differential Invariants: The moving frame method provides a direct route to the classification of joint invariants and joint differential invariants[15,39]. Further developments appear in Boutin's thesis[5,6].

Equivalence, Symmetry and Rigidity: The fundamental differential invariants, as specified by the recurrence formulae, serve to parametrize the signature manifold associated with a given submanifold. For example the Euclidean signature of a plane curve is the curve parametrized by the first two differential invariants κ, κ_s. The signature completely solves the basic equivalence problem: Two submanifolds be mapped to each other by a group transformation if and only if they have the same signature[15,10,35]. Extensions to noise-resistant joint invariant signatures are extensively developed[39]. Applications include general rigidity theorems for submanifolds under group actions[15].

Calculus of Variations: Most modern physical theories begin by postulating a symmetry group and then formulating field equations based on a

group-invariant variational principle. As first recognized by Sophus Lie[29], every invariant variational problem can be written in terms of the differential invariants of the symmetry group. The associated Euler-Lagrange equations inherit the symmetry group of the variational problem, and so can also be written in terms of the differential invariants. The moving frame constructions were applied to establish a general group-invariant formula that enables one to directly construct the Euler-Lagrange equations from the invariant form of the variational problem[26,27]. These results are based on the invariant variational bicomplex construction and the resulting recurrence formulae. An alternative foundation of the subject, based on a new approach to symmetry reduction of exterior differential systems and variational problems, can be found in Itskov[20].

Classical Invariant Theory: The moving frame theory was applied to produce new, practical algorithms for solving the basic symmetry and equivalence problems of univariate polynomials (binary forms) that form the foundation of classical invariant theory[35,3,24]. An early version of the required signature was based on a fortuitous connection with a Cartan equivalence problem in the calculus of variations[33,34]. Extensions to polynomials in several variables can be found in Kogan's thesis[24].

Poisson Geometry and Solitons: Moving frames have been used to classify the differential invariants of projective curves and surfaces, and applied to generate integrable Poisson flows in soliton theory[31]. A similar construction for space curves under the conformal group appears in Marí Beffa[30].

Computer Vision: Earlier work on applications of the Cartan moving frame theory can be found in Faugeras[13]. The general characterization of submanifolds via their differential invariant signatures was applied to the problem of object recognition and symmetry detection in digital images[10]. Boutin[5,7] applies moving frame methods to the problems of polygon recognition and symmetry detection. Extensions to projective actions appear in the recent thesis of Hann[19].

Numerical Methods and Geometric Integration: The approximation of higher order differential invariants by joint invariants underlies the formulation of fully invariant finite difference numerical schemes[9,10,4,5]. Applications of moving frames to the construction of invariant numerical algorithms and the theory of geometric integration[8,21] are under development[40,28].

Infinite-dimensional Pseudo-groups: The moving frame algorithm has been extended to several examples of infinite-dimensional pseudo-group actions[14]. However, a full, rigorous foundation for the theory has yet to be completed. Once completed, the theory will produce pseudo-group versions of all of the preceding applications.

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CRITICAL POINT THEORY AND HAMILTONIAN DYNAMICS AROUND CRITICAL ELEMENTS

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We give a few examples of how, using the knowledge available on the geometry of Hamiltonian dynamics with symmetry, standard critical point theory can be adapted to this setup in order to obtain predictions on the existence of various dynamical elements and, moreover, it can be used to provide estimates on the number of these solutions. The proofs of these results, as well as additional information, can be found in [29,27]

1 Critical point theory in the Hamiltonian symmetric context

The use of the symmetries of a mechanical system, propitiated by all the founders of the field (especially H. Poincaré and E. Nöther) has produced very important fruits. In our discussion we will significantly use the philosophy advocated by S. Smale [34] who proposed the study of the topology of the spaces resulting by quotienting the level sets of the existing conserved quantities (energy and momentum) by the relevant group actions. As we will see, a thorough study of this reduced kinematics is capable to produce valuable dynamical information.

To be more specific, we will look for relative critical elements of a symmetric Hamiltonian system by reducing their search to the existence of critical points of an appropriate function to one of the reduced spaces proposed by Smale. Once we have formulated the problem in such a way we will use some methods of critical point theory, mainly Lusternik-Schnirelman and Morse theories, in order to solve it.

As to the Lusternik-Schnirelman categorical approach we will use mainly three ideas:

- Let M be a compact G–manifold, with G a compact Lie group. Any G–invariant smooth function $f \in C^\infty(M)^G$ has at least $\mathrm{Cat}(M/G)$ critical orbits [39].

- Let M be a compact G–manifold, with G a Lie group acting properly on M such that the isotropy subgroup of each point $m \in M$ is a finite subgroup of G. Let ω be a symplectic G–invariant form defined on M. Let $H \subset G$ be a Lie subgroup of G and $N \subset M$ be a H–invariant closed submanifold of M such that for any $n \in N$ we have that

$$(T_nN)^\omega = \mathfrak{g} \cdot n \qquad \text{and} \qquad T_nN \cap \mathfrak{g} \cdot m = \mathfrak{h} \cdot n, \tag{1.1}$$

  where $\mathfrak{g} \cdot n = \left\{\xi_M(n) := \frac{d}{dt}\big|_{t=0} \exp t\xi \cdot n \mid \xi \in \mathfrak{g}\right\}$ denotes the tangent space at $n \in N \subset M$ of the G–orbit $G \cdot n = \{g \cdot n \mid g \in G\}$ and $(T_nN)^\omega$ denotes the ω–orthogonal space of T_nN in T_nM. Then, there is a cohomology class $\theta \in H^2(N/H; \mathbb{R})$ such that $\theta^k \neq 0$, where $k = \frac{1}{2}(\dim N - \dim H)$ [39]. This property amounts to a lower bound for the value of the Lusternik-Schnirelman category of a symplectic quotient [19] resulting from a locally free action.

- Let G be a compact Lie group that contains a maximal torus T and that acts linearly on the vector space V. Suppose that the vector subspace V^T of T–fixed vectors on V is trivial, that is, $V^T = \{0\}$, then any G–invariant function defined in the unit sphere of V (the unit sphere of V is defined with the aid of a G–invariant norm on it) has at least

$$\frac{\dim V}{2(1 + \dim G - \dim T)} = \frac{\dim V}{2(1 + \dim G - \mathrm{rank}\, G)}$$

  critical orbits [2].

Regarding Morse theory, the symplectically reduced spaces where we will search for critical points are exactly those studied by Kirwan in [12], hence we will be able to use in a straightforward manner her results.

2 Relative equilibria and periodic orbits around non degenerate critical points

We illustrate the techniques discussed in the previous section with a simple application whose proof can be found in [29].

Theorem 2.1 *Let $(V, \omega, h, G, \mathbf{J})$ be a Hamiltonian G–vector space, with G a compact Lie group. Suppose that $h(0) = 0$, $\mathbf{d}h(0) = 0$, and the quadratic form $Q := \mathbf{d}^2h(0)$ on V is definite. Let $\xi \in \mathfrak{g}$ be such that the quadratic form*

$\mathbf{d}^2\mathbf{J}^\xi(0)$ *is non degenerate. Then, for each energy value* ϵ *small enough, there are at least*

$$\mathrm{Cat}\left(h^{-1}(\epsilon)/G^\xi\right) = \mathrm{Cat}\left(Q^{-1}(\epsilon)/G^\xi\right) \tag{2.2}$$

distinct relative equilibria in $h^{-1}(\epsilon)$ *whose velocities are (real) multiples of* ξ. *The symbol* $G^\xi := \{g \in G \mid \mathrm{Ad}_g\, \xi = \xi\}$ *denotes the adjoint isotropy of the element* $\xi \in \mathfrak{g}$ *and* Cat *is the Lusternik–Schnirelman category. If the compact Lie group* G^ξ *has a maximal torus* T^ξ *such that the set* V^{T^ξ} *of* T^ξ*–fixed vectors on* V *is trivial, that is,* $V^{T^\xi} = \{0\}$, *then there are at least*

$$\frac{\dim V}{2(1 + \dim G^\xi - \mathrm{rank}\, G^\xi)}. \tag{2.3}$$

distinct relative equilibria in $h^{-1}(\epsilon)$ *whose velocities are (real) multiples of* ξ.

If the G-symmetry in the preceeding theorem is given by the S^1-action resulting of putting the system in normal form [37] we obtain as a corollary a generalization to the symmetric framework of the Weinstein-Moser Theorem on the existence of periodic orbits around a stable equilibrium. More explicitely, we have [24,2,29]:

Corollary 2.2 (Equivariant Weinstein–Moser theorem) *Let* $(V, \omega, h, G, \mathbf{J})$ *be a Hamiltonian* G*–vector space, with* G *a compact Lie group, such that* $h(0) = 0$, $\mathbf{d}h(0) = 0$, *and the infinitesimally symplectic linear map* $A := DX_h(0)$ *is non singular and has* $\pm i\nu_\circ$ *in its spectrum. Let* $U_{\nu_\circ}$ *be the resonance space of* A *with primitive period* $T_{\nu_\circ}$ *and consider the canonical* S^1*–action on* $U_{\nu_\circ}$ *defined by the restriction to this symplectic subspace of the flow of* A. *Let* $H \subset G \times S^1$ *be an isotropy subgroup of the* $G \times S^1$*–action on* $U_{\nu_\circ}$. *If the restriction of* $Q := \mathbf{d}^2h(0)$ *to* $(U_{\nu_\circ})^H$ *is a definite quadratic form then, for any sufficiently small* $\epsilon > 0$ *there are at least*

$$\frac{\dim(U_{\nu_\circ})^H}{2(1 + \dim(N(H)/H) - \mathrm{rank}\,(\mathrm{N(H)/H}))} \tag{2.4}$$

geometrically distinct periodic orbits in each energy level $h^{-1}(\epsilon)$ *whose periods tend to* $T_{\nu_\circ}$ *as* ϵ *tends to zero and whose isotropy subgroups include* H.

3 Relative periodic orbits

This section is a brief account of the main result in [27]. A genuine generalization of the Weinstein-Moser Theorem in the symmetric framework should be able to predict *relative periodic orbits* (RPOs) in the neighboring energy-momentum level sets of a stable *relative equilibrium*. Moreover, it should be able to give an estimate on the number of these solutions for any prescribed

spatiotemporal structure, that is, for any isotropy subgroup of the natural $G \times S^1$-action present in the problem. In this direction we have:

Theorem 3.1 *Let $(V, \omega, h, G, \mathbf{J} : V \to \mathfrak{g}^*)$ be a Hamiltonian system with symmetry, with V a vector space, and G a compact positive dimensional Lie group that acts on V in a linear and canonical fashion. Suppose that $h(0) = 0$, $\mathbf{d}h(0) = 0$ (that is, the Hamiltonian vector field X_h has an equilibrium at the origin) and that the linear Hamiltonian vector field $A := DX_h(0)$ is non degenerate and contains $\pm i\nu_\circ$ in its spectrum. Let $U_{\nu_\circ}$ be the resonance space of A with primitive period $T_{\nu_\circ} := \frac{2\pi}{\nu_\circ}$. Consider the $G \times S^1$-action on $U_{\nu_\circ}$, where the S^1-action is induced by the semisimple part of A, and the Lie group G acts simply on $U_{\nu_\circ}$ Let $H = \{(k, \theta_H(k)) \mid k \in K \subset G\} \subset G \times S^1$ be an isotropy subgroup of the $G \times S^1$-action on $U_{\nu_\circ}$ with temporal character θ_H, temporal velocity $\rho_H \in \mathfrak{k}^*$, and such that the quadratic form Q^H on the H-fixed point space $U^H_{\nu_\circ}$ defined by*

$$Q^H(v) := \frac{1}{2}\mathbf{d}^2 h(0)(v, v), \qquad v \in U^H_{\nu_\circ}$$

is definite. Then, for any $\chi_\circ \in (\mathfrak{k}^\circ)^K$ for which $\mathbf{J}|^{-1}_{(U_{\nu_\circ})_H}\left(\chi_\circ - \frac{1}{\nu_\circ}\rho_H\right) \cap Q_H^{-1}(1)$ is non empty ($Q_H := Q^H|_{(U_{\nu_\circ})_H}$) there exists an open neighborhood $V_{\chi_\circ}$ of $\chi_\circ$ in $(\mathfrak{k}^\circ)^K$ such that for any $\chi \in V_{\chi_\circ}$, the intersection $\mathbf{J}|^{-1}_{(U_{\nu_\circ})_H}\left(\chi - \frac{1}{\nu_\circ}\rho_H\right) \cap Q_H^{-1}(1)$ is a submanifold of $(U_{\nu_\circ})_H$ of dimension $\dim U^H_{\nu_\circ} - \dim N(H)/H$. Suppose that the following two generic hypotheses hold:

(H1) *The restriction $h|_{U^H_{\nu_\circ}}$ of the Hamiltonian h to the fixed point subspace $U^H_{\nu_\circ}$ is not radial with respect to the norm associated to Q^H.*

(H2) *Let $h_k(v) := \frac{1}{k!}\mathbf{d}^k h(0)\left(v^{(k)}\right)$, $v \in U^H_{\nu_\circ}$ be the first non radial term in the Taylor expansion of $h|_{U^H_{\nu_\circ}}$ around zero. We will assume that $k \geq 4$ and that the restrictions of h_k to the submanifolds $\mathbf{J}|^{-1}_{(U_{\nu_\circ})_H}\left(\chi - \frac{1}{\nu_\circ}\rho_H\right) \cap Q_H^{-1}(1)$, with $\chi \in V_{\chi_\circ}$, are Morse–Bott functions with respect to the $(N_G(K)_{\rho_H} \cap N_G(K)_\chi) \times S^1$-action.*

Then, for any $\epsilon > 0$ close enough to zero, $\chi \in V_{\chi_\circ}$, and $\lambda := \Xi^*(\chi - \frac{1}{\nu_\circ}\rho_H, \frac{1}{\nu_\circ})$, there are at least

$$\max\left[\alpha, \chi_E\left(\mathbf{E}\mathbf{J}^{-1}_{L_H}(\lambda)\right)^{L_\lambda}\right] \tag{3.5}$$

distinct relative periodic orbits of X_h with energy ϵ, momentum $\epsilon(\chi - \frac{1}{\nu_\circ}\rho_H) \in \mathfrak{g}^*$, isotropy subgroup H, and relative period close to $T_{\nu_\circ}$. Here

$$\alpha \;=\; \frac{1}{2}\left(\dim U^H_{\nu_\circ} - \dim N_G(K) - \dim\left(N_G(K)_{\rho_H} \cap N_G(K)_\chi\right) + 2\dim K\right) \;.$$

The symbol $\chi_E\left(\mathbf{EJ}_{L_H}^{-1}(\lambda)\right)^{L_\lambda}$ *denotes the* L_λ*–Euler characteristic of* $\mathbf{EJ}_{L_H}^{-1}(\lambda)$ *(which in this case equals the standard Euler characteristic of the symplectic quotient* $\chi_E(\mathbf{EJ}_{L_H}^{-1}(\lambda)/L_\lambda)$.

The first part of estimate 3.5 is a product of the use of the Lusternik-Schnirelman category on the problem, while the second one is obviously coming out of symplectic Morse theory.

4 Further generalizations and future directions

For the sake of simplicity, in the preceeding two sections we chose to present results that predict periodic and relative periodic solutions, as well as relative equilibria, in the neighborhood of a symmetric *equilibrium on a symplectic vector space*. This situation can be generalized to *relative equilibria on a manifold* using the so called *reconstruction equations* [26,32,30] on the normal coordinates of Marle, Guillemin, and Sternberg [16,17,10].

All the situations considered in the statements presented above involve non degeneracy hypotheses of one form or another. These restrictions allow to preserve the Hamiltonian structure through all the reductions carried out to solve the problem. The degenerate situations involve completely different techniques and require a much more analytic study. In other words, the geometry of the problem is still important but its understanding is not sufficient to fully describe the dynamics associated to it. The reader is encouraged to check with [29] for results similar to the ones presented here, in the presence of degeneracies.

The previous statement is especially relevant when attempting to understand the bifurcation phenomena in Hamiltonian symmetric systems. For instance, the appearance of RPOs in a Hamiltonian symmetric Hopf bifurcation is a natural problem to study where some answers have been given [6]. Nevertheless, the full picture is far from being understood.

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COMPUTING INVARIANT MANIFOLDS OF PERTURBED DYNAMICAL SYSTEMS

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We study perturbed dynamical systems defined through an m–dimensional vector field. In this paper we illustrate how the computation of different normal forms of these systems leads to the calculation of different invariant manifolds of them.

1 Introduction

Let us consider the ordinary differential equation

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(\mathbf{x}(t);\varepsilon;\mathbf{c}) = \mathbf{F}_0(\mathbf{x}(t);\mathbf{c}) + \sum_{i=1}^{L} \frac{\varepsilon^i}{i!}\mathbf{F}_i(\mathbf{x}(t);\mathbf{c}), \tag{1}$$

where t represents the independent variable, $\mathbf{x} \in \mathbf{R}^m$, $\mathbf{c} \in \mathbf{R}^m$ are parameters associated to the problem, ε stands for a dimensionless small parameter and for $0 \leq i \leq L$, $\mathbf{F}_i$ is a vector field with m components defined on an open set $\Omega \subseteq \mathbf{R}^m$. Our goal is to find out periodic orbits and other invariant manifolds of (1).

We use the methodology presented in the work by Palacián and Yanguas [14] for the computation of asymptotic invariant manifolds associated to (1). It is a generalization of previous work for vector fields [12] and Hamiltonians [11] of polynomial type. The main idea is to construct generalized normal forms, i.e. different formal changes of variables which lead to different systems of differential equations (the normal forms). For this purpose we make use of Lie transformations [8]. Thus, (1) is transformed into different systems, each of them enjoying a different symmetry up to a certain order of approximation. As an example of the calculation of normal forms of non–polynomial vector fields we mention the work by Cushman [4] on perturbed Keplerian systems.

Once an asymptotic symmetry is determined, a reduction map allows us to pass to the reduced system, which is defined in the orbit space. Its dimension is $s < m$ and it is parameterized by the invariants associated to the reduction map. A non–degenerate p–dimensional (with $p \leq s$) invariant set of the reduced system is transformed, asymptotically, into a $(p + m - s)$–dimensional invariant set of the departure system. Moreover, these invariant sets are calcu-

lated explicitly (up to an adequate order) by inverting the Lie transformations involved in the computation of the normal forms.

Making use of the Splitting Lemma (see the paper by Gaeta[7] and references therein) it is readily proven that the transformed vector field can be split into two subsystems defined on two different invariant spaces. One of the subsystems, the so–called reduced system, contains the fundamental dynamics of the departure system.

The paper has four sections. Sec. 2 recalls the Generalized Normal Form Theorem. In Sec. 3 we roughly describe the geometrical aspects of the reduction process. These two subjects are more deeply studied in the work by the authors[14]. In Sec. 4 the procedure is applied to the Lorenz system and to the Hamiltonian vector field defined by the restricted three body problem, so as to characterize new periodic orbits and $2D$ invariant manifolds.

2 Extending Symmetries of the Unperturbed Part

We recall the Generalized Normal Form Theorem we make use of in order to calculate formal symmetries of $\mathbf{F}$. Its proof is given in a previous paper[12].

Theorem 2.1 *Let $M \geq 1$ be given, let $\{\mathcal{P}_i\}_{i=0}^{M}$, $\{\mathcal{Q}_i\}_{i=1}^{M}$ and $\{\mathcal{R}_i\}_{i=1}^{M}$ be sequences of vector spaces of analytic functions in $\mathbf{x} \in \mathbf{R}^n$ defined on a common domain Ω in $\mathbf{R}^n$ and let $\mathbf{T} \equiv \mathbf{T}(\mathbf{x})$ be a vector field in some $\{\mathcal{P}_i\}_{i=0}^{M}$ with the following properties:*

i) $\mathcal{Q}_i \subseteq \mathcal{P}_i$, $i = 1, \ldots, M$;

ii) $\mathbf{F}_i \in \mathcal{P}_i$, $i = 0, 1, \ldots, M$;

iii) $[\mathcal{P}_i\,, \mathcal{R}_j\,] \subseteq \mathcal{P}_{i+j}$, $i + j = 1, \ldots, M$;

iv) for any $\mathbf{D} \in \mathcal{P}_i$, $i = 1, \ldots, M$, one can find $\mathbf{E} \in \mathcal{Q}_i$, $\mathbf{K} \in \mathcal{R}_i$ such that

$$\mathbf{E} = \mathbf{D} + [\mathbf{F}_0\,, \mathbf{K}] \quad \textit{and} \quad [\mathbf{E}\,, \mathbf{T}] = \mathbf{0}.$$

Then, there is an analytic vector field $\mathbf{W}$,

$$\mathbf{W}(\mathbf{x}; \varepsilon) = \sum_{i=0}^{M-1} \frac{\varepsilon^i}{i!} \mathbf{W}_{i+1}(\mathbf{x}),$$

with $\mathbf{W}_i \in \mathcal{R}_i$, $i = 1, \ldots, M$, such that the change of variables $\mathbf{x} = \mathbf{X}(\mathbf{y}; \varepsilon)$ is the general solution of the initial value problem

$$\frac{d\mathbf{x}}{d\varepsilon} = \frac{\partial \mathbf{W}}{\partial \mathbf{x}}(\mathbf{x}; \varepsilon), \qquad \mathbf{x}(0) = \mathbf{y},$$

and transforms the convergent vector field

$$\mathbf{F}(\mathbf{x};\varepsilon) = \sum_{i=0}^{\infty} \frac{\varepsilon^i}{i!} \mathbf{F}_i(\mathbf{x}),$$

to the convergent vector field

$$\mathbf{G}(\mathbf{y};\varepsilon) = \sum_{i=0}^{M} \frac{\varepsilon^i}{i!} \mathbf{G}_i(\mathbf{y}) + \mathcal{O}(\varepsilon^{M+1}),$$

with $\mathbf{G}_i \in \mathcal{Q}_i$ *and* $[\mathbf{G}_i, \mathbf{T}] = \mathbf{0}$, $i = 1, \ldots, M$. *Besides, if* $[\mathbf{F}_0, \mathbf{T}] = \mathbf{0}$ *then* $\mathbf{T} \equiv \mathbf{T}(\mathbf{y})$ *is a formal symmetry of* $\mathbf{G}$.

Note that the difference between this result and General Perturbation Theorem given by Meyer [8] is that here we introduce $\mathbf{T}$ and require that functions $\mathbf{E} \in \mathcal{Q}_i$ satisfy $[\mathbf{E}, \mathbf{T}] = \mathbf{0}$. This vector field $\mathbf{T}$ gives us the freedom to compute different asymptotic symmetries, different normal forms. Let us call $\mathbf{H}(\mathbf{y};\varepsilon)$ the reduced system, i.e., the truncation at order M of $\mathbf{G}$:

$$\frac{d\mathbf{y}}{dt} = \mathbf{H}(\mathbf{y};\varepsilon) = \sum_{i=0}^{M} \frac{\varepsilon^i}{i!} \mathbf{G}_i(\mathbf{y}). \tag{2}$$

This vector field is a generalized normal form of the original vector field (1). Note that the number of generalized normal forms one can calculate depends on the different Lie transformations of $\mathbf{F}(\mathbf{x};\varepsilon)$ one executes, or in other words, on the independent symmetries $\mathbf{T}$ corresponding to $\mathbf{F}_0$.

This transformation is not convergent, in general, but we do not address this subject here. Some results about convergence of transformations based on normal form techniques can be consulted in the reference by Walcher [16], as well as in the book by Cicogna and Gaeta [3].

3 Reduction

From a geometrical point of view, the consequence of introducing a symmetry by making use of Theorem 2.1 is that the dimension of the phase space is reduced from m to s (s denoting the number of functionally–independent first integrals associated to $\mathbf{T}$).

Related to the one–parameter group of symmetries introduced through the Lie transformation there is an $(m-s)$–dimensional Lie group $G_{\mathbf{T}}$, such that $\mathbf{H}$ is $G_{\mathbf{T}}$–equivariant, that is, fixed $\varepsilon > 0$, for any $\mathbf{y} \in \mathbf{R}^m$ and any $g \in G_{\mathbf{T}}$, $\mathbf{H}(\mathbf{y},\varepsilon) = \mathbf{H}(g\,\mathbf{y},\varepsilon)$. Schwartz [15] proved that for any $G_{\mathbf{T}}$–equivariant vector

field, there is a set of smooth functions $\varphi_i(\mathbf{y}), i = 1, \ldots, r$ defined on a domain $\Omega \subseteq \mathbf{R}^m$ such that any $G_\mathbf{T}$–equivariant smooth function defined in Ω can be written as a $\mathcal{C}^\infty(\Omega)$–function of $\varphi_i(\mathbf{y})$. Besides, these functions correspond to the r linearly–independent first integrals of the system $d\mathbf{y}(t)/dt = \mathbf{T}(\mathbf{y}(t))$, from which $1 \leq s \leq r$ are functionally independent.

We define the reduction map as the surjective map:

$$\begin{aligned} \pi_\mathbf{T} : \Omega \subseteq \mathbf{R}^m &\longrightarrow \quad \mathbf{R}^m/G_\mathbf{T} \\ \mathbf{y} &\mapsto \mathbf{p} = \{\varphi_1, \ldots, \varphi_r\}. \end{aligned}$$

Now, related to the reduction map $\pi_\mathbf{T}$ and the vector field (2), there is a phase space defined as the s–dimensional quotient space $\mathbf{R}^m/G_\mathbf{T}$ (which is a semialgebraic manifold, the so–called orbit space, see details in the book by Cushman and Bates [5]). Henceforth, the φ_i are also called the invariants of the reduction process. We choose a set of co–ordinates on $G_\mathbf{T}$ to make the reduction explicit. Denoting $\mathbf{q} = \{\vartheta_1, \ldots, \vartheta_{m-s}\}$ the flow on $G_\mathbf{T}$ is indeed the time evolution of the variables $\vartheta_i \in G_\mathbf{T}$. We have the following result due to Gaeta [7].

Theorem 3.1 *Splitting Lemma. Given the generalized normal form system (2) with* $\mathbf{H}$ *a smooth function of* ε *and* $\mathbf{y}$ *defined on* $\Omega \subseteq \mathbf{R}^m$, *it can be transformed into a triangular system as:*

$$\begin{aligned} \frac{d\mathbf{p}(t)}{dt} &= \mathbf{a}(\mathbf{p}(t); \varepsilon), \\ \frac{d\mathbf{q}(t)}{dt} &= \mathbf{b}(\mathbf{q}(t), \mathbf{p}(t); \varepsilon), \end{aligned} \tag{3}$$

$\mathbf{a}$ *and* $\mathbf{b}$ *being smooth functions obtained constructively from* $\mathbf{H}$.

The first equation of (3) depends exclusively on the φ_i, it is named the reduced system and is defined on $\mathbf{R}^m/G_\mathbf{T}$, whereas the second equation of (3) is defined on the Lie group $G_\mathbf{T}$. The vector field $\mathbf{a}$ is constructed using the identity $d\mathbf{p}(t)/dt = (\partial \mathbf{p}/\partial \mathbf{y})\ \mathbf{H}(\mathbf{y}; \varepsilon)$ taking into account that the right–hand part of this equation can be expressed completely in terms of $\mathbf{p}$. Thus, we identify $\mathbf{a}(\mathbf{p}; \varepsilon) = (\partial \mathbf{p}/\partial \mathbf{y})\ \mathbf{H}(\mathbf{y}; \varepsilon)$. The construction of $\mathbf{b}$ is performed once the co–ordinates $\mathbf{q}$ have been calculated.

In order to go through the details of the reduction process, the explicit calculation of the invariant manifolds of the original system from the study of the reduced system and estimates of the error committed by the application of Theorem 2.1, the reader can consult the work by the authors [14].

The clue of this method is that the type of invariant manifold of the original system determined through a certain normal form depends on the type of vector field $\mathbf{T}$ and consequently on its invariants or co–ordinates in $G_\mathbf{T}$.

4 Examples of Application

We choose two well–known deeply studied problems. The first one, the Lorenz system, which is polynomial and disipative. The second one, the $3D$ restricted three–body problem: a non–polynomial Hamiltonian system.

4.1 The Lorenz System

The Lorenz system has been treated from the point of view of generalized normal forms by the authors in a previous paper [13]. The classical definition of the system is as follows:

$$\frac{dx}{dt} = 10(y-x), \quad \frac{dy}{dt} = rx - y - xz, \quad \frac{dz}{dt} = xy - \frac{8}{3}z, \tag{4}$$

where r is a positive parameter and t represents the time variable. The linear part of the system is defined by the matrix:

$$A = \begin{pmatrix} -10 & 10 & 0 \\ r & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{pmatrix}.$$

The eigenvalues of A are non–resonant. Thus, the Poincaré–Dulac normal form is linear and no qualitative information about the original system can be inferred from it. Using generalized normal forms we have found some invariant sets. In order to calculate generalized normal forms of (4) and compute thereafter invariant manifolds of it, we have to choose a vector field $\mathbf{T}$, which in the polynomial case is: $\mathbf{T}(\mathbf{x}) = T\,\mathbf{x}$, in such a way that T commutes with A and the normal form be nontrivial. So, according to the work by Yanguas [17] and taking into account that A is diagonalizable with distinct eigenvalues:

$$T = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix}.$$

Thus, different choices of t_i lead to different invariant sets of (4). For instance, in Fig. 1 we present two invariant sets for different values of T.

4.2 The Restricted Three–Body Problem

The Hamiltonian is defined as follows:

$$\mathcal{H} = -(-x_2\,y_1 + x_1\,y_2) + \frac{1}{2}(y_1^2 + y_2^2 + y_3^2) - \frac{\mu}{\sqrt{(-1+\mu+x_1)^2 + x_2^2 + x_3^2}} - \frac{1-\mu}{\sqrt{(\mu+x_1)^2 + x_2^2 + x_3^2}},$$

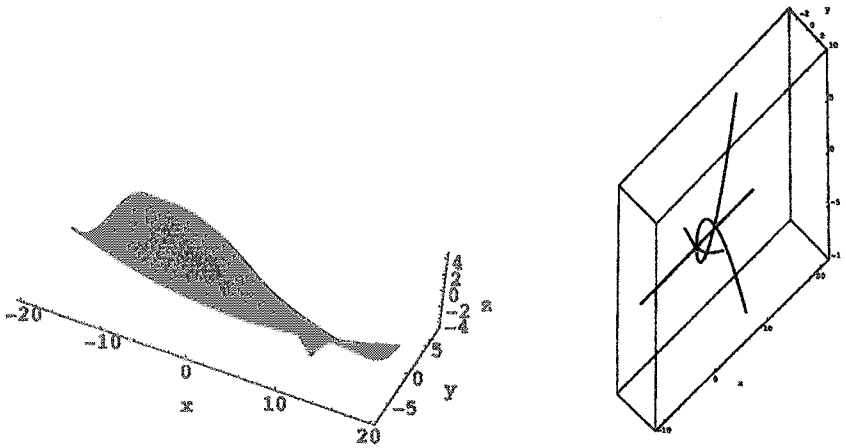

Figure 1: On the left the stable manifold of the origin is depicted. The values of the coefficients of matrix T are: $t_3 = 1, t_1 = t_2 = 0$ and $r = 3.205189$. On the right we present some one–dimensional invariant sets for $t_3 = t_2 = t_1/2$.

where $\mu > 0$ represents the mass of the first primary, whereas $1 - \mu > 0$ corresponds to the mass of the other primary. The coordinates of the infinitesimal particle in a rotating coordinate system are (x_1, x_2, x_3) and the momenta (or velocities): (y_1, y_2, y_3). The position of the first primary is: $(1 - \mu, 0, 0)$ and the second primary is placed at $(-\mu, 0, 0)$.

Here we only consider the so–called lunar problem[9], where the infinitesimal particle is very near one of the primaries. We put one primary at the origin, rescale variables and Hamiltonian by introducing the small positive parameter ε as a measure of the distance from the infinitesimal particle to the first primary and perform a series expansion in ε up to order 5. Thus, we reach:

$$\mathcal{H} = \frac{1}{2\,\varepsilon^3}\left(\frac{-2}{\sqrt{x_1^2+x_2^2+x_3^2}} + y_1^2 + y_2^2 + y_3^2\right) - (-x_2\,y_1 + x_1\,y_2)$$
$$+\frac{\varepsilon^3}{2}\,\mu\,(-2\,x_1^2 + x_2^2 + x_3^2) + \frac{\varepsilon^5}{2}\,\mu\,(1-\mu)^{1/3}\,x_1\,[-2\,x_1^2 + 3\,(x_2^2 + x_3^2)]$$

and we can consider $\mathcal{H}$ as a perturbed Keplerian Hamiltonian. In this way we can apply the techniques used to deal with this kind of systems. For instance, we express $\mathcal{H}$ in Delaunay variables (see the paper by Deprit[6] and references therein): $\mathcal{H}(\ell, g, h, L, G, H; \varepsilon)$ in order to calculate normal forms. Now, we have different possibilities to extend (formally) symmetries of the unperturbed system to the whole perturbed one. As L and H are integrals of the unper-

turbed part, we can either choose the energy of the unperturbed system: L, as the integral to be extended, the third component of the angular momentum vector: H, or both. We summarize these possibilities in the following commutative diagram:

$$\begin{array}{ccc} \mathbf{R}^6 & \xrightarrow{L \text{ is an integral}} & S_L^2 \times S_L^2 \\ \Big\downarrow{\scriptstyle H \text{ is an integral}} & & \Big\downarrow{\scriptstyle H \text{ is an integral}} \\ \mathbf{R}^6/(S^1 \times S^1)_H & \xrightarrow{L \text{ is an integral}} & \mathcal{T}_{L,H} \end{array}$$

With the two first reductions, families of periodic orbits are determined in two degrees of freedom, whereas from the double reduction, families of $2D$ invariant tori are calculated in one degree of freedom.

Making L an integral of the perturbed transformed system up to a certain order is just performing a Delaunay normalization [6]. In this case we have reached order eight in the procedure. Fixing $L > 0$ the reduced phase space is four–dimensional: the product of the two–spheres:

$$S_L^2 \times S_L^2 = \{(\mathbf{a}, \mathbf{b}) \in \mathbf{R}^6 \mid a_1^2 + a_2^2 + a_3^2 = L^2, \quad b_1^2 + b_2^2 + b_3^2 = L^2\}.$$

We have calculated the equilibria on $S_L^2 \times S_L^2$. They correspond to periodic orbits of the original system parameterized by L. In fact, we found the periodic orbits given by Moser [10], which are rectilinear and circular equatorial.

Normalizing the argument of the node: h, we make that H be an integral of the transformed perturbed system up to order nine, in this case. The reduced phase space is also four–dimensional:

$$\mathbf{R}^6/(S^1 \times S^1)_H = \{(c_1, \ldots, c_6) \in \mathbf{R}^6 \mid c_1 c_4 = c_2^2 + c_5^2, \quad c_5 = H\}.$$

The equilibria we found on $\mathbf{R}^6/(S^1 \times S^1)_H$ correspond to equatorial periodic orbits, which are the ones found by Arenstorf [1] for the planar case.

For details on the these two reductions, the reader can consult the work by Cushman [4].

Extending both integrals: L and H, we reach the two–dimensional reduced phase space defined by:

$$\mathcal{T}_{L,H} = \{(\tau_1, \tau_2, \tau_3) \in \mathbf{R}^3 \mid \tau_2^2 + \tau_3^2 = [L^2 - (\tau_1 - H)^2] \times [L^2 - (\tau_1 + H)^2]\},$$

where $0 \leq |H| \leq L$, $L > 0$, $\tau_2, \tau_3 \in [H^2 - L^2, L^2 - H^2]$ and $\tau_1 \in [H - L, L - H]$.

The equilibria of the doubly reduced system correspond to quasiperiodic (stable) circular equatorial orbits, stable rectilinear in the axis Oz (due to Belbruno[2]) and unstable rectilinear equatorial. Some examples are in Fig. 2.

Figure 2: On the left: equatorial quasiperiodic orbits. On the right: circular quasiperiodic orbits.

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PERIODIC SOLUTIONS FOR RESONANT NONLINEAR PDES

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In this review we study the periodic behaviour of some nonlinear PDEs looked upon as infinite dimensional dynamical systems. We briefly illustrate the origins and the motivations of this kind of problems and some of the lines along with they have been tackled. We explain then our contribution, which consists in a suitable combination of bifurcation and averaging techniques, leading to very simplified proofs and to results for completely resonant systems. Application to nonlinear string equation, and to some higher dimensional equations are given.

1 Introduction

The main results we are presenting in this proceeding are the following: We obtain existence of families of small amplitude periodic orbits for a class of equations whose linear part is completely resonant, and we apply it to a nonlinear string equation; for such equation we also give a stability result of Nekhoroshev type for the fundamental mode. We then develop a version of the previous abstract result for the partially resonant case, in order to deal with higher dimensional equations, applying it to a nonlinear plate equation in any spatial dimension.

For what concerns the existence of periodic solutions, the problem takes origin from the very classical and old question of the continuation of periodic orbits from linear to nonlinear systems, which goes back to Poincaré and Lyapunov, and which in turn arises from the study of the small oscillations about an elliptic equilibrium point.

In the finite dimensional setting, the Lyapunov center theorem states that: Given n oscillators, with a nonlinear coupling which is of higher order near the origin, such that there exists a first integral, if the frequency of an oscillator is not in resonance with all the other frequencies, then the normal mode related to that oscillator can be continued to the nonlinear system. If the system is Hamiltonian, which provides the existence of the first integral, given by the Hamiltonian itself, and all the frequencies are non-resonant between them, then all the n linear periodic solutions, actually the normal modes, locally persist in the full system, with small deformations for their trajectories and their frequencies.

It is then natural to ask for extensions of this result to resonant systems

and to infinite dimensional systems.

We remark at this point that when one looks for quasi periodic solutions, which is the subject of the KAM theory, he has to manage the problem of the small divisors: But if the system is infinite dimensional, this difficulty is already present in the search for periodic orbits.

For resonant but finite dimensional systems, there are results by Weinstein[17] and Moser[12]: They state that if the system is Hamiltonian, and the Hamiltonian function, near the origin, is positive definite on the subspace spanned by the possibly resonant modes, then there are n periodic solutions for every small energy, where n is again the number of degrees of freedom.

On the other hand, for infinite but non-resonant systems, there have been various extensions providing persistence of both periodic and quasi-periodic solutions, via KAM methods (see for example the works by Kuksin[9,10], Pöschel[14] and Wayne[16]) and by a new approach based on Lyapunov-Schmidt decomposition and Nash-Moser techniques (see Craig & Wayne [7,8] and Bourgain[5]). The latter approach proved to allow to deal with partial resonances, as those arising from periodic boundary conditions or from a spatial dimension of the problem larger than one.

Very few results, instead, exist for infinite dimensional completely resonant systems, apart from Bourgain[6] and Lidskii & Shulman[11], and those obtained by a variational approach (see for example[15]), which are of a quite different nature.

2 Resonant systems

The first result we present is of an abstract type, and is concerned with an infinite system of ODEs of the form

$$u_{tt} + Au = f(u) \,, \tag{1}$$

in a suitable Hilbert space, where A is a positive self-adjoint operator with completely resonant pure point spectrum, and f is a nonlinearity whose first non-vanishing term $f^{(0)}$ in the Taylor expansion is homogeneous at least of order three; we also denote by $\langle f^{(0)} \rangle$ the average, with respect to the linear flow Φ^t, of $f^{(0)}$, namely:

$$\langle f^{(0)} \rangle (x) := \frac{1}{2\pi} \int_0^{2\pi} \Phi^{-t} \left[f^{(0)} \left(\Phi^t(x) \right) \right] dt \,.$$

Theorem 1. *Consider the setting above: Then, associated to every non-degenerate zero of the map $A + \langle f^{(0)} \rangle$ there is a Cantor family of periodic*

solutions with small energy, whose trajectories and frequencies are close to that of the linear evolution of such a zero.

The above statement can be made more precise. Furthermore, when the nonlinearity is of gradient type, i.e. $f(u) = \nabla F(u)$, it is possible to give a formulation in terms of critical points of $\langle F^{(0)} \rangle|_{E^{-1}(c)}$ instead of zeros of $A + \langle f^{(0)} \rangle$ (where it holds $Au = \nabla E(u)$); the forthcoming scheme of proof refers to this situation — where moreover, for simplicity, we also assume $f^{(0)}$ to be homogeneous of order three. See the original paper[4] for further details.

Proof As in the work by Craig & Wayne[7], the main scheme of the proof is based on a Lyapunov-Schmidt decomposition, but our attention is concentrated on the kernel equation, which in our case is infinite dimensional due to the full resonance of the problem. It is solved by exploiting a natural isomorphism between such a kernel and the configuration space of the system, and by making use of averaging theory.

First of all we define a set of 'good frequencies' W_γ, as those which are sufficiently non resonant with respect to the spectrum of A: They will be the frequencies of the solutions we are able to find. Denoting by $\{k_j^2\}$ the spectrum of A, with k_j integers, we set:

$$W_\gamma := \{\omega \in \mathbf{R} \quad \text{s.t.} \quad |\omega l - k_j| > \frac{\gamma}{l} \quad \text{whenever} \quad l \neq k_j\} \; ; \tag{2}$$

this set turns out to be uncountable, with all the integers as accumulation points.

We then re-scale time to look for solutions with fixed period: Fix $\omega \in W_\gamma$, and consider the operator $L_\omega q := \omega^2 q_{tt} + Aq$; we look for solutions of the equation

$$L_\omega q = f^{(0)}(q) + f^{(1)}(q) \; , \tag{3}$$

with q in the Hilbert space $\mathcal{H}_s \subset H^1(\mathbf{R}/2\pi\mathbf{Z}, \ell_s^2)$ of the functions of the type

$$q(t) = \sum_{j \in \mathbf{N}} \left\{ q_{0j} + \sum_{l \geq 1} q_{lj} \cos(lt) \right\} e_j \; ,$$

(e_j being the eigenvectors of A relative to k_j^2), with the norm

$$\|q\|^2 := \sum_{j \in \mathbf{N}} j^{2s} \left\{ 2q_{0j}^2 + \sum_{l \geq 1} q_{lj}^2 (1 + l^2) \right\} \; .$$

Solutions of 3 define periodic solutions of 1 with frequency ω.

We use a Lyapunov–Schmidt decomposition on the kernel of L_1: remark that such a kernel is constituted by all the solution of the linearization of 1,

since all the eigenvalues of A are integers. Let $K := ker L_1$ and $R := K^{\perp}$, and denote by $Q : \mathcal{H}_s \to K$ an $P : \mathcal{H}_s \to R$ the projectors on these space. Decompose q as follows:

$$q = \epsilon v + \epsilon^3 w \ , \quad v = Qv \ , \quad w = Qw \ , \quad \epsilon \in \mathbf{R}$$

and let impose on v the normalization condition $||v|| = 1$; we also make the ansatz

$$\omega^2 = 1 + \beta\epsilon^2 \tag{4}$$

with β to be determined. Inserting in equation 3 and decomposing it on K and R we have

$$L_\omega w = Pf^{(0)}(v) + \frac{1}{\epsilon^3} P\left[f^{(0)}(\epsilon v + \epsilon^3 w) - f^{(0)}(\epsilon v) + f^{(1)}(\epsilon v + \epsilon^3 w)\right] \ , \ (P)$$

$$-\beta A v = Qf^{(0)}(v) + \frac{1}{\epsilon^3} Q\left[f^{(0)}(\epsilon v + \epsilon^3 w) - f^{(0)}(\epsilon v) + f^{(1)}(\epsilon v + \epsilon^3 w)\right] \ , \ (Q)$$

where the parts in square brackets turn out to be higher order corrections.

The first step is the solution of the equation on the range, which involves the small denominators: Since $\omega \in W_\gamma$, the eigenvalues $\lambda_{jl} := (-l\omega + k_j)(l\omega + k_j)$ of L_ω are such that

$$|\lambda_{jl}| \geq \frac{\gamma}{l}\omega l = \gamma\omega \ ,$$

and therefore L_ω is invertible on its range, with a bounded inverse, whose norm is uniformly bounded for $\omega \in W_\gamma$. Thus one can apply L_ω^{-1} to the equation (P) and apply to it implicit function theorem in order to construct w as a function of $v \in K$, of $\omega \in W_\gamma$ and of ϵ small enough. Remark that $w(v, \epsilon)$ is close to $L_\omega^{-1} P f^{(0)}(v)$.

The second step is the solution of the kernel equation (equation (Q)). To this end remember that the (Q) equation is a perturbation of the equation

$$-\beta A v = Qf^{(0)}(v) \ , \tag{5}$$

so the idea is to solve 5 and then to use implicit function theorem in order to solve equation (Q).

The main remark is that equation (Q) is equivalent to an equation on the configuration space $\mathcal{Q}$. In fact K is generated by $\{\cos(k_j t)e_j\}$, and therefore the isomorphism between these spaces is simply given by

$$\begin{aligned} I : \mathcal{Q} &\to K \\ z &\to I(z) \equiv q \qquad \text{with } q(t) := \Phi^t z \end{aligned}$$

where $\Phi^t z \equiv \Phi^t(\sum_j z_j e_j) := \sum_j z_j \cos(k_j t) e_j$ is the flow of the linearized system in the space $\mathcal{Q}$. Using such an isomorphism it is possible to pull back equation 5 to the configuration space; in particular one can easily verify by explicit computation that such a pull back takes the form

$$-\beta \nabla E(z) = \nabla \langle F^{(0)} \rangle(z) \ , \tag{6}$$

where $\langle F^{(0)} \rangle : \mathcal{Q} \to \mathbf{R}$ is the average of $F^{(0)}$, defined by

$$\langle F^{(0)} \rangle(z) := \frac{1}{T} \int_0^T F^{(0)}(\Phi^t z) dt \ ,$$

and the function E is the harmonic energy defined by

$$E(z) := \frac{1}{2} \sum_j k_j^2 z_j^2 \ .$$

Then equation 6 is equivalent to the equation for the critical points of the restriction of $F^{(0)}$ to suitable level surface E, which is a sphere. Here β appears as a Lagrange multiplier. Let now z_0 be a non-degenerate critical point of $\mathcal{F} := F^{(0)}\big|_{E^{-1}(c)}$, then by implicit function theorem it can be continued to a solution of the Q equation.

Finally one has to satisfy the equation 4, which is now an equation for ϵ. It can be reduced to a fixed point equation that can be solved by contraction mapping principle provided $\omega \in W_\gamma$ is sufficiently close to 1.

With minor changes it is possible to show that the frequency actually can be chosen near n if the linear flow of the critical point has frequency n (in particular, in the above procedure one has to consider the decomposition on the kernel of L_n instead of that of L_1).

Remark *One of the main differencies between the resonant and the non-resonant cases is that, in the latter one, periodic orbits of the linearized system are isolated, while for the resonant situations they constitues linear subspaces. So, switching on the nonlinearity, in one case there is clear continuation techniques involved, while in the resonant setting we first have to choose which linear orbit will be continued exploiting the nonlinearity to break the described nondegeneracy.*

2.1 Nonlinear string equations

The attention is now addressed to the system which actually constitutes one of the fundamental and motivating examples for considering the abstract setting cited above, namely the nonlinear string equation. The delicate point in using

the general theorem consists in looking for the above mentioned critical points and in checking their non-degeneracy, which is not trivial at all.

Define

$$\xi_n(x,t) := \frac{V}{n}\left[\mathrm{sn}\left(n\mathsf{w}(x+t)|\mathsf{m}\right) - \mathrm{sn}\left(n\mathsf{w}(-x+t)|\mathsf{m}\right)\right]$$

where sn is the elliptic sine, w and m are constants defined by the boundary condition stated below, and V is normalization constant; we then have the following

Theorem 2. *Consider a nonlinear string equation of the form* $u_{tt} - u_{xx} = \pm u^3 + \mathcal{O}(u^4)$, *with Dirichlet boundary conditions on* $[0,\pi]$. *Then there are countably many Cantor families, in the energy* ϵ, *of periodic solutions, whose trajectories and frequencies are close to those of* $\epsilon\xi_n(x,t)$. *Moreover, the family with lowest frequency exhibit a Nekhoroshev type stability, i.e. solutions starting* $\mathcal{O}(\epsilon^2)$ *close to it, remain* $\mathcal{O}(\epsilon^2)$ *close to it for times of order* $\mathcal{O}(\exp(1/\epsilon))$.

This result can be found in [4] concerning the existence part, and in [13] for the stability result and for various lemmas used also in [4].

To obtain the proof of the theorem one has to put the system in normal form, explicitly calculating the average $\langle F^{(0)}\rangle$; one has to find all the critical points required by theorem 1, and to verify their non degeneracy. For the application of the theory of [2] ensuring the stability property, it must be checked that only one of these critical points is an extremum.

3 Plate equations in higher dimension

In this section we consider a class of systems in higher space dimension, which typically exhibit resonances even when the corresponding one dimensional version is non resonant: This typically reflects the symmetries of the spatial domain of the equation.

So, consider the following nonlinear plate equation

$$u_{tt} + \Delta\Delta u + mu = \pm u^3 + \mathcal{O}(u^5) \tag{7}$$

with Dirichlet boundary conditions in a n–dimensional cube. We use the notation

$$\xi_{(j_1,\ldots,j_k)}(x_1,\ldots,x_k,t) := \sin(j_1x_1)\cdots\sin(j_kx_k)\cos(\omega_{(j_1,\ldots,j_k)}t)$$

for some solution of the linear system, around which we look for the periodic orbits.

Theorem 3. *Consider the system 7. Then, for any* n *and for* m *belonging to an uncountable and dense subset of the interval* $[0, n^2/4]$, *there exist a Cantor*

family of periodic solutions with trajectory and frequency close to that of the fundamental linear mode, $\xi_{(1,\dots,1)}$.

Concerning the modes which exibits resonances, the theory permits to have a clear picture of the situation: the following statement must be considered as an example of the results we are able to obtain.

Theorem 4. *Consider the system 7. Then, for* $n = 2$, *there are Cantor families of periodic orbits with trajectories and frequencies near those of* $\xi_{(1,2)}$, $\xi_{(2,1)}$, $\xi_{(1,2)} \pm \xi_{(2,1)}$ *with* m *is in a dense subset of* $[0, 25/4]$, *and near those of* $\xi_{(1,7)}$, $\xi_{(7,1)}$, $\xi_{(5,5)}$, $\xi_{(1,7)} \pm \xi_{(7,1)}$, $\xi_{(1,7)} \pm \xi_{(5,5)}$, $\xi_{(5,5)} \pm \xi_{(7,1)}$, *with* m *is in a dense subset of* $[0, 25/2]$.

The main scheme for the proofs is again a Lyapunov-Schmidt decomposition: We choose the kernel of L_ω, where ω is the frequency (possibly multiple) of the linear solution we try to continue.

This theorem (see [3] for full details) is a sort of intermediate result between those of [4] and [1]: In fact, when the parameter m of the system is not zero, the linear part is no more completely resonant, but due to the multiplicity of some eigenvalues, it is neither completely non-resonant.

So, as in Lyapunov theorem, one must ask that the frequencies belonging to the resonant block are sufficiently non-resonant with the other, through a Diophantine condition: This is necessary to solve the range equation in a way similar to that showed in the proof of theorem 1. This is the most delicate point, because one wants to verify this condition for a sufficiently large set of the parameter m.

For the existence of periodic solutions near the fundamental mode (theorem 3) this is enough, since such a mode is always simple, and thus the kernel equation is trivially one dimensional. Concerning the higher modes (theorem 4), one exploit again the isomorphism between the range and an appropriate configuration space; then, since the resonance is finite — since the multiplicity of the frequency is finite — it is quite easy to calculate explicitly the average of the perturbation and to look for its critical points as in the proof of the first theorem. In this way it is possible to obtain the exact number of solutions and their localization.

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A SYMMETRIC NORMAL FORM FOR THE FERMI PASTA ULAM CHAIN

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The Fermi Pasta Ulam chain with periodic boundary conditions admits discrete and continuous symmetries. These symmetries allow one to formulate important restrictions on the Birkhoff normal form of this Hamiltonian system. We derive integrability properties and KAM statements. Hence the combination of symmetry and resonance in the periodic Fermi Pasta Ulam chain explains its quasiperiodic behaviour. This article contains a summary of the results obtained in references [11] and [12]

1 Introduction

The Fermi Pasta Ulam chain with periodic boundary conditions is a model for point masses moving on a circle with nonlinear forces acting between the nearest neighbours. Let us set $q_j \in \mathbb{R}$ to be the position of the j-th particle $(j = 1, \ldots, n)$ with respect to a certain reference position on the circle. The space of positions $q = (q_1, \ldots, q_n)$ of the particles in the chain is $\mathbb{R}^n$. The space of positions and conjugate momenta is the cotangent bundle $T^*\mathbb{R}^n$ of $\mathbb{R}^n$, the elements of which are denoted $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$. $T^*\mathbb{R}^n$ is a symplectic manifold, endowed with the symplectic form $dq \wedge dp = \sum_{j=1}^{n} dq_j \wedge dp_j$. Any smooth function $H : T^*\mathbb{R}^n \to \mathbb{R}$ now induces the Hamiltonian vector field X_H given by the defining relation $\iota_{X_H}(dq \wedge dp) = dH$. In other words, we have the system of ordinary differential equations $\dot{q}_j = \frac{\partial H}{\partial p_j}$, $\dot{p}_j = -\frac{\partial H}{\partial q_j}$. The periodic FPU chain with n particles is the special Hamiltonian system on $T^*\mathbb{R}^n$ corresponding to the real analytic Hamiltonian

$$H = \sum_{j \in \mathbb{Z}/n\mathbb{Z}} \frac{1}{2} p_j^2 + V(q_{j+1} - q_j) \ , \tag{1.1}$$

in which $V : \mathbb{R} \to \mathbb{R}$ is a potential energy function of the form

$$V(x) = \frac{1}{2!}x^2 + \frac{\alpha}{3!}x^3 + \frac{\beta}{4!}x^4 + \ldots \ . \tag{1.2}$$

The $\alpha, \beta, \ldots$ are real parameters measuring the nonlinearity in the forces between the particles in the chain.

Numerically, the FPU system was first studied by E. Fermi, J. Pasta and S. Ulam [5]. These authors used the chain to model a nonlinear string. They expected that in the presence of small nonlinearities, the chain would show ergodic behaviour, meaning that almost all orbits densely fill up an energy levelset of the Hamiltonian. Ergodicity would eventually lead to an equal distribution of energy between the various Fourier modes of the system, a concept called *thermalisation*. FPU's nowadays famous numerical experiment was intended to investigate at what time scale thermalisation would take place. The result was astonishing: it turned out that there was no sign of thermalisation at all. Putting initially all the energy in one Fourier mode, they observed that this energy was shared by only a few other modes, the remaining modes were hardly excited. Additionally, within a not too long time the system returned close to its initial state.

In 1965 an article of Zabuski and Kruskal [15] appeared. These authors considered the Korteweg-de Vries equation as a continuum limit of the FPU chain and numerically found the first indications for the stable behaviour of solitary waves. We now know that the Korteweg-de Vries equation is integrable [9]. This clearly suggests an explanation for FPU's observations, although the relation between the FPU chain and its infinite dimensional limits has never been understood.

Another, possibly correct explanation for the quasiperiodic behaviour of the FPU system, is based on the Kolmogorov-Arnol'd-Moser theorem. As is well-known [2], the solutions of an n degrees of freedom Liouville integrable Hamiltonian system are constrained to move on n-dimensional tori and are not at all ergodic but periodic and quasiperiodic. The KAM theorem states that most of the invariant tori of this integrable system persist under small Hamiltonian perturbations, if the unperturbed integrable system satisfies a certain nondegeneracy condition. This nondegeneracy condition states that the frequency map, which assigns to each n-dimensional invariant torus of the integrable system the n-dimensional vector of frequencies of the (quasi)periodic motion on this torus, be a local diffeomorphism. Although several authors, starting with Izrailev and Chirikov [7], have stated that the KAM theorem explains the observations of the FPU experiment, it is still completely unclear how the FPU system can be seen as a perturbation of a nondegenerate integrable system. One could only view it as a perturbation of a harmonic oscillator, but the frequency map of the harmonic oscillator is constant and hence degenerate. This gap in the theory was recently mentioned again in the review article of Ford [6] and the book of Weissert [14].

The only serious attempt to overcome this problem was made in 1971 in

a paper by Nishida [8]. Unlike us, this author considers an FPU chain with fixed endpoints. Under a rather strong nonresonance condition on the linear frequencies of his system, he shows that there is a nonlinear symplectic near-identity transformation of phase space, the 'Birkhoff transformation', with the following property: written out in the new coordinates, the Hamiltonian function of the FPU chain turns out to be a perturbation of a nondegenerate integrable system. And hence the KAM theorem can be applied. The weakness of this argument lies of course in the fact that the linear frequencies actually do not satisfy the imposed nonresonance condition.

Sanders [13] does a similar thing for FPU chains with periodic boundary conditions and an odd number of particles. Assuming a nonresonance condition, he observes that the normal form is again integrable, but he does not verify the KAM nondegeneracy condition.

In this short paper I shall give a summary of the results that F. Verhulst and I [11] [12] obtained in trying to generalise the work of Nishida and Sanders. In particular, we computed all the lower order resonance relations in the eigenvalues of the linearized FPU chain. And secondly, we exploited the discrete symmetries of the periodic FPU chain to show that its Birkhoff normal form has some very special properties. In a lot of the cases, one can actually prove that it is integrable or even satisfies the KAM nondegeneracy condition. We do not impose any nonresonance condition. For more details concerning the calculation, the reader should of course consult the original references [11] and [12].

2 The linear system

One would like to view the solutions of the equations induced by (1.1) as a superposition of sine and consine wave forms. Therefore, one usually applies a Fourier transformation $(q,p) \mapsto (\bar{q},\bar{p})$. The new coordinates $(\bar{q},\bar{p})$ are called 'phonons' or 'quasi-particles'. The transformation to phonons is a linear symplectic point transformation. We omit the transformation matrix here. See [11] or [12] for the exact formulas. The transformation is such that when written out in phonon-coordinates, the FPU Hamiltonian reads

$$H = \sum_{j=1}^{n-1} \frac{1}{2}(\bar{p}_j^2 + \omega_j^2 \bar{q}_j^2) + H_3(\bar{q}_1, \ldots, \bar{q}_{n-1}) + H_4(\bar{q}_1, \ldots, \bar{q}_{n-1}) + \ldots \tag{2.1}$$

where H_k $(k = 2, 3, \ldots)$ denotes the k-th order part of H. For $j = 1, \ldots, n-1$, the numbers ω_j are the eigenvalues of the linear periodic FPU problem:

$$\omega_j := 2\sin(\frac{j\pi}{n}) \ . \tag{2.2}$$

Expressions for H_3 and H_4 in terms of the $\bar{q}_j$ can be found in the literature, cf. [10]. We do not repeat them.

Note that the new Hamiltonian has $n-1$ degrees of freedom instead of n, because simultaneously with introducing the phonons, we divided out the symmetry induced by the flow of the total momentum $p_1 + \ldots + p_n$, which is a constant of motion. More details can be found in [11]. The Hamiltonian (2.1) on $T^*\mathbb{R}^{n-1}$ represents the periodic FPU system from which the centre of mass motion has been eliminated.

Since $\omega_j^2 > 0$ $(1 \leq j \leq n-1)$, using the Morse-Lemma [1] we conclude that the level sets of H are $2n-3$ dimensional spheres around the origin of $T^*\mathbb{R}^{n-1}$. Since H is a constant of motion for the flow of X_H, the origin is a stable stationary point for the system induced by (2.1). It corresponds to an equidistant configuration of the particles.

3 Birkhoff normalisation

From (2.1) we see that the solutions of the linearized FPU system are simply superpositions of pulsating wave forms. But in the full nonlinear system the Fourier modes can exchange energy. We shall study this much harder system using Birkhoff normalisation, hoping to be able to apply KAM theory and bifurcation methods.

The setting of normalisation is the following. Let P_k be the space of homogeneous polynomials of degree k in the canonical variables $(\bar{q}_1, \ldots, \bar{q}_{n-1}, \bar{p}_1, \ldots, \bar{p}_{n-1})$. The space of all convergent power series without linear part is denoted $P \subset \bigoplus_{k>2} P_k$. P is a Lie-algebra under the usual Poisson bracket $\{f, g\} = dq \wedge dp(X_f, X_g)$. Finally, for each $f \in P$ one defines the adjoint operator $\mathrm{ad}_f : P \to P$ which maps $\mathrm{ad}_f : g \mapsto \{f, g\}$. Note that when $\mathrm{ad}_f(g) = \{f, g\} = 0$, then the flows of X_f and X_g commute. The following result is well-known:

Theorem 3.1 (Birkhoff) *Let $r > 2$ be a given natural number. Assume that $H = \sum_{k=2}^{\infty} H_k \in P$ is such that for each $3 \leq k \leq r$, $\mathrm{ad}_{H_2} : P_k \to P_k$ is semisimple, i.e. complex diagonalisable. Then there is an open neighborhood $U \in T^*\mathbb{R}^{n-1}$ of the origin and an analytic symplectic diffeomorphism Ψ : $U \to \Psi(U) \subset T^*\mathbb{R}^{n-1}$ such that $\Psi(0) = 0$, $D\Psi(0) = Id$ and $\overline{H} := H \circ \Psi =$*

$\sum_{k=2}^{\infty} \overline{H}_k \in P$ *has the properties that* $\overline{H}_2 = H_2$ *and* $ad_{H_2}(\overline{H_k}) = \{H_2, \overline{H}_k\} = 0$ *for all* $2 \leq k \leq r$.

The transformed Hamiltonian $\overline{H}$ is called a Birkhoff normal form for H of order r. It can be determined following a rather explicit procedure, which the reader can find in [3] and [11]. It is usually impossible to push r to infinity.

In the case of the periodic FPU Hamiltonian (2.1), $\mathrm{ad}_{H_2} : P_k \to P_k$ is indeed semisimple and its eigenvalues are the numbers

$$\sum_{j=1}^{n-1} i\omega_j(\eta_j - \theta_j) \quad . \tag{3.1}$$

where η, θ are $n-1$-dimensional multi-indices with the property that $|\eta| + |\theta| := \sum_{j=1}^{n-1} |\eta_j| + |\theta_j| = k$.

It is important to study the kernel of ad_{H_2} because this kernel contains all possible normal forms of the FPU chain. Therefore we wonder whether some of the eigenvalues (3.1) are zero. In this case we speak of 'resonance'. There are some trivial resonance relations: choose for instance $\eta_j = \theta_j$. Note also that from (2.2) it follows that $\omega_j = \omega_{n-j}$. This yields even more rather trivial 1:1 resonances.

Are there more resonance relations?

Nishida [8] and Sanders [13] had to make the assumption that weren't any, although in fact there are. Using Galois theory, we calculated all the resonance relations for which $|\eta| + |\theta| = 3, 4$. We got substantial help from Frits Beukers at this point. To give the reader some feeling for the type of resonance relations we found, we give two of them here:

$$2\sin\frac{\pi}{6} - \sin\frac{3\pi}{6} = 0 \quad \text{and} \quad \sin\frac{5\pi}{30} + \sin\frac{13\pi}{30} - \sin\frac{7\pi}{30} - \sin\frac{9\pi}{30} = 0 \ .$$

The first one is rather trivial, but the second is not. Nishida [8] and Sanders [13] were worried that this type of nontrivial resonances could spoil their normal form results. It is therefore very surprising that these resonances turn out to be completely harmless. This is caused by discrete symmetries.

4 Discrete symmetry

Consider the following maps in the space of positions of the particles:

$$T : \partial_{q_j} \mapsto \partial_{q_{j-1}} \quad \text{and} \quad S : \partial_{q_j} \mapsto -\partial_{q_{n-j}} \tag{4.1}$$

T and S represent permutations that rotate and flip the particles respectively. They can be extended to symplectic point transformations on $T^*\mathbb{R}^n$. These point transformations, which we shall also denote T and S, leave the Hamiltonian of the periodic FPU problem (1.1) invariant: $T^*H := H \circ T = H$ and $S^*H := H \circ S = H$. This implies that the Hamiltonian vector field X_H induced by H is equivariant under T and S. Therefore, T and S are called discrete symmetries of H. The group $\langle T, S \rangle$ generated by T and S is isomorphic to the n-th dihedral group, the symmetry group of the n-gon.

Finally, S and T project to symmetries of the reduced Hamiltonian (2.1).

5 The symmetric normal form

A crucial observation, which was brought to our attention by J.J. Duistermaat, is that one can construct Birkhoff normal forms, that respect these symmetries. In other words, one can choose to make normal forms that have the same symmetries as the Hamiltonian one started out with. For a proof of this statement, the reader can consult [3].

In the case of the FPU chain with periodic boundary conditions, this means that the nonquadratic terms of the normal form $H_2 + \overline{H}_3 + \overline{H}_4 + \ldots$ satisfy

$$\mathrm{ad}_{H_2}(\overline{H}_k) = 0 \ , \ (T^* - \mathrm{Id})(\overline{H}_k) = 0 \text{ and } (S^* - \mathrm{Id})(\overline{H}_k) = 0 \ . \tag{5.1}$$

In other words, $\overline{H}_k$ is in the joint kernel of the linear operators $\mathrm{ad}_{H_2}, T^* - \mathrm{Id}$ and $S^* - \mathrm{Id}$ and it is our task to determine this joint kernel. This is the computation that constitutes the main part of [12]. Note that we already know what the kernel of ad_{H_2} is, as we have already calculated all the resonances. In the computation of the joint kernel one uses the fact that ad_{H_2} and $T^* - \mathrm{Id}$ commute to search inside the kernel of ad_{H_2} for degenerate directions of $T^* - \mathrm{Id}$. The invariance under S^* is then used to refine the results. I list the most important conclusions here:

1 The set of homogeneous third order polynomials that satisfy (5.1) is $\{0\}$. So $\overline{H}_3 = 0$ for the periodic FPU chain, independent of the number of particles n and the resonances in the eigenvalues.

2 If the number of particles n is odd, then the truncated normal form $H_2 + \overline{H}_4$ is Liouville integrable. The integrals are quadratic and constitute global action-angle coordinates. This is true for every homogeneous fourth order polynomial that satisfies (5.1), so we conclude it for the FPU chain without even calculating its normal form.

3 If the number of particles n is even, then we give a lot of integrals of the truncated normal form $H_2 + \overline{H}_4$, again without computing the normal form. But we can not prove Liouville integrability this way.

Just like Nishida [8] and Sanders [13], we explicitly calculated the normal form of the so-called β-chain, for which the nonlinearity-coefficient α is zero. This yields even more information:

4 If n is odd, then the truncated normal form $H_2 + \overline{H}_4$ of the β-chain satisfies the KAM nondegeneracy condition. Since the original system can be seen as a perturbation of this truncated normal form, the conclusions of the KAM theorem hold. We proved that most low energy solutions of the odd β-chain lie on tori.

5 If n is even, then the truncated normal form $H_2 + \overline{H}_4$ of the β-chain turns out to be Liouville integrable too. We have no explanation of this in terms of symmetries. It is very difficult to check the KAM condition in this case, since we have no expression for the action-angle coordinates. In fact, there are strong indications that global action-angle coordinates do not exist in this case. On the other hand, the integrability of the normal form makes one suspect that the KAM condition should actually hold.

Although we have a lot of results, there are obviously still many open questions.

6 Conclusion

The results of this paper might not immediately apply to the experiment of Fermi, Pasta and Ulam. First of all, these authors did not study a periodic chain. Secondly, it is not certain that the normal form approximation is still valid at the energy level they chose in their experiment.

But still, the special combination of the eigenvalues and discrete symmetries of the FPU problem might be the true reason for the observations that Fermi, Pasta and Ulam did.

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ONE-DIMENSIONAL INFINITE SYMMETRIES, BOUNDARY CONDITIONS, AND LOCAL CONSERVATION LAWS

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We will consider Lagrangian differential equations of dimensions two and higher whose symmetry group generators contain an arbitrary function of one independent variable. We will study the relationship between symmetries, boundary conditions and local conservation laws, and discuss comparison with the Second Noether Theorem. We will demonstrate the conclusions on the example of the equation of nonstationary transonic gas flow.

1 Introduction

In this paper we will discuss a problem somewhat intermediate between the First and the Second Noether Theorems [1]. We consider a differential equation admitting a symmetry algebra with operators depending on an arbitrary function of one of the independent variables along with its derivatives (one-dimensional infinite symmetries). We will be interested in the equations corresponding to a well-defined action functional where symmetries of the functional (variational or Noether symmetries) are at the same time the symmetries of our differential system (see e.g. [2]). We will show that the considered one-dimensional infinite symmetries do not lead to an infinite number of conservation laws. We will demonstrate that some conservation laws corresponding to given symmetries are possible and analyze the role of the boundary conditions.

2 Noether operator identity. First and Second Noether Theorems

Let

$$S = \int L(x^i, u, u_i, \cdots) d^n x \tag{1}$$

be the action functional, where L is the density of Lagrangian, $x^i = (t, x, y, \ldots,)$, $i = 1, \ldots, n$ are independent and u are dependent variables, $u_i \equiv \partial u / \partial x^i$. Then

$$E(L) \equiv \omega(x, u, u_i, u_{ij} \ldots) = 0 \tag{2}$$

is the equation of motion, where E is the Euler-Lagrange operator

$$E = \frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + \sum_{i \le j} D_i D_j \frac{\partial}{\partial u_{ij}} + \cdots . \tag{3}$$

We consider an infinitesimal transformation of the form (see, e.g. [2] or [3]:

$$\begin{aligned} x'^i &= x^i + \varepsilon \xi^i(x, u, u_j, \ldots) + O(\varepsilon^2) \\ u' &= u + \varepsilon \eta(x, u, u_j, \ldots) + O(\varepsilon^2), \end{aligned} \tag{4}$$

where ε is a small parameter. The operator corresponding to this transformation is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u} + \zeta^i \frac{\partial}{\partial u_i} + \cdots . \tag{5}$$

In place of X we will consider a canonical operator X_α with $\alpha = \eta - \xi^i u_i$:

$$X_\alpha = \alpha \frac{\partial}{\partial u} + (D_i \alpha) \frac{\partial}{\partial u_i} + \sum_{i \le j} (D_i D_j \alpha) \frac{\partial}{\partial u_{ij}} + \cdots . \tag{6}$$

In the future we will make use of the Noether identity (see [4] (or [5]), or [6] for a version used here):

$$X_\alpha = \alpha E + D_i R_{\alpha i}. \tag{7}$$

where

$$\begin{aligned} R_{\alpha i} &= \alpha \frac{\partial}{\partial u_i} + \left\{ \sum_{k \ge i} (D_k \alpha) - \alpha \sum_{k \le i} D_k \right\} \frac{\partial}{\partial u_{ik}} \\ &+ \left\{ \sum_{k \ge j \ge i} (D_j D_k \alpha) - \sum_{k \le i \le j} (D_j \alpha) D_k + \alpha \sum_{j \le k \le i} D_j D_k \right\} \frac{\partial}{\partial u_{ijk}} + \cdots . \end{aligned} \tag{8}$$

The variation of the functional S under a transformation (4) with the symmetry vector α and operator X_α is

$$\delta S = \int [X_\alpha L + D_i(\xi^i L)] d^n x . \tag{9}$$

Let α be a variational (Noether) symmetry:

$$X_\alpha L = D_i M_i . \tag{10}$$

The application of Noether identity (7) to L gives:

$$X_\alpha L = \alpha \omega + D_i(R_{\alpha i} L) . \tag{11}$$

Combining (11) with (10) we will get:

$$D_i(M_i - R_{\alpha i} L) = \alpha \omega , \tag{12}$$

which means that the total divergence of some vector vanishes on the solution manifold ($\omega = 0,\ D_i \omega = 0, \ldots$):

$$D_i N_i \doteq 0 . \tag{13}$$

Thus, any variational symmetry α (in case of a finite group) leads to a corresponding conservation law (12) - First Noether Theorem [1].

Consider now a case when the symmetry vector α is of the form

$$\alpha = ap(x) + b_i D_i p(x) + c_{ij} D_i D_j p(x) + \ldots \tag{14}$$

where *p(x)* is an *arbitrary function* (infinite group). We will apply the Noether identity (7) to L

$$X_a L = \alpha^b E^b(L) + D_i(R_{ai} L), \tag{15}$$

and transform the RHS of (15):

$$\begin{aligned} \alpha^b E^b(L) &= (a^b p + b_i^b D_i p + c_{ij}^b D_i D_j p + \ldots)\omega^b \\ &= p a^b \omega^b + D_i(b_i^b \omega^b p) - p D_i(b_i^b p) + \ldots \\ &= p(x)\{a^b \omega^b - D_i(b_i^b \omega^b) + D_i D_j(c_{ij}^b \omega^b) + \ldots\} + D_i(b_i^b \omega^b p + \ldots). \end{aligned} \tag{16}$$

Thus,

$$\alpha^b E^b(L) = p(x)\tilde{a}(\omega, D_i\omega, \ldots) + D_i K_i , \tag{17}$$

where $K_i \sim p(x)$, and

$$\tilde{a} = a^b \omega^b - D_i(b_i^b \omega^b) + D_i D_j(c_{ij}^b \omega^b) + \ldots . \tag{18}$$

Since α is a variational symmetry using (10), (15) and (17) we obtain:

$$D_i M_i = p(x)\tilde{a} + D_i K_i + D_i(R_{\alpha i} L), \tag{19}$$

or

$$D_i T_i = p(x)\tilde{a} , \tag{20}$$

where $T_i = M_i - K_i + R_{\alpha i} \sim \mathrm{p(x)}$. Equation (20) has a form of continuity equation, but in general it does not imply a conservation law. Let us integrate (20) over the whole space in R^n and use the Gauss theorem:

$$\int_V D_i T_i dV = \int_{\partial V} T_i n_i d\sigma = \int_V p(x)\tilde{a}\, dV . \tag{21}$$

Consider the following cases:

I. $p = p(x_1, x_2, \ldots, x_n)$. *p(x) is an arbitrary function of all variables in* R^n.

In this case it is possible to choose the function $p(x)$ such that it vanishes on the

boundary together with its derivatives:

$$p(x)\Big|_{\partial V} = D_i p(x)\Big|_{\partial V} = \ldots = 0. \tag{22}$$

Then $T_i\Big|_{\partial V} = 0$. From equation (21) using arbitrariness of V we can write

$$D_i T_i = 0 \tag{23}$$

and from (20) we conclude

$$\tilde{a}(\omega, D_i\omega, \ldots) = 0. \tag{24}$$

Therefore

$$a^b\omega^b - D_i(b_i^b\omega^b) + D_iD_j(c_{ij}^b\omega^b) + \ldots = 0, \tag{25}$$

where $a^b, b_i^b, c_{ij}^b, \ldots$ are components of the symmetry α (14). Equation (25) expresses the Second Noether Theorem: in case of an infinite variational symmetry group not all equations of the original differential system are independent [1].

II. $p = p(x_1, x_2, \ldots, x_k), x \in R^n, \ k < n.$

p(x) is an arbitrary function of some base variables.

In this case we generally cannot choose *p(x)* such that it vanishes on the boundary together with its derivatives. Therefore $T_i \sim p(x)$ will not, in general, vanish on the boundary ($T_i \not\to 0$ as $x_i \to \partial V$) and we can no longer conclude that $\tilde{a} = 0$. Further we will consider a case $k = 1$ when the generators of the variational symmetry group contain an arbitrary function of one of independent variables.

3 Infinite symmetry algebra with arbitrary function of time

Let $x^i = (x, y, t)$. Consider a Noether symmetry α of the form:

$$\alpha = a\gamma(t) + b\gamma'(t) + c\gamma''(t) + \ldots + h\gamma^{(l)}(t). \tag{26}$$

We have:

$$\delta S = \int \delta L d^n x = \int X_\alpha L d^n x = \int D_i M_i d^n x = 0. \tag{27}$$

Therefore

$$\int M_x \Big|_{x\to-\infty}^{x\to\infty} d^{n-1}x + \int M_y \Big|_{y\to-\infty}^{y\to\infty} d^{n-1}x + \int M_t \Big|_{t\to-\infty}^{t\to\infty} d^{n-1}x = 0 \tag{28}$$

and it follows that the following conditions must be satisfied

$$M_x(x,u,\ldots)\Big|_{x\to-\infty}^{x\to\infty} = M_y(x,u,\ldots)\Big|_{y\to-\infty}^{y\to\infty} = M_t(x,u,\ldots)\Big|_{t\to-\infty}^{t\to\infty} = 0. \quad (29)$$

We will call equations (29) *Noether boundary conditions*. Equations (29) are usually satisfied for a "regular" asymptotic behavior: $u, u_i \to 0$ as $x \to \pm\infty$, or for periodic conditions.

Let us consider now another type of boundary conditions related to the existence of local conservation laws. Integrating equation (12) over the whole space (x, y) and restricting ourselves to the solution manifold we will get:

$$\iint dxdy\left[D_x(M_x - R_{\alpha x}L) + D_y(M_y - R_{\alpha y}L) + D_t(M_t - R_{\alpha t}L)\right] \doteq 0 \quad (30)$$

or

$$\iint dxdy D_t(M_t - R_{\alpha t}L) \doteq \int dy(R_{\alpha x}L - M_x)\Big|_{x\to\partial D} + \int dx(R_{\alpha y}L - M_y)\Big|_{y\to\partial D}. \quad (31)$$

Since α is a Noether symmetry, we can apply the Noether boundary condition (29). Requiring that the LHS of (31) vanish (Noether conservation laws), we will obtain:

$$R_{\alpha x}L\Big|_{x\to\partial D} = R_{\alpha y}L\Big|_{y\to\partial D} = 0. \quad (32)$$

We will call equations (32) *"strict" boundary conditions*. Thus in order for the system to possess (Noether) local conserved quantities, the conditions (29) and (32) have to be satisfied. Note that in case $L = L(x^i, u, u_i)$ (second order PDE's) strict boundary conditions (32) take a simple form:

$$\alpha\frac{\partial L}{\partial u_x}\Big|_{x\to\partial D} = \alpha\frac{\partial L}{\partial u_y}\Big|_{y\to\partial D} = 0. \quad (33)$$

If both Noether (29) and strict boundary conditions (32) are satisfied the Noether conservation laws will read:

$$\iint dxdy D_t(M_t - R_{\alpha t}L) \doteq 0. \quad (34)$$

Let us demonstrate that the equation (34) does not lead to an infinite number of conservation laws on the example of the case $L = L(x^i, u, u_i)$. Writing M_t as

$$M_t = A\gamma(t) + B\gamma'(t) + C\gamma''(t) + \ldots + H\gamma^{(l)}(t) \quad (35)$$

and using equation (26), we will obtain:

$$D_t \iint dxdy\left[\gamma(A - a\frac{\partial L}{\partial u_t}) + \gamma'(B - b\frac{\partial L}{\partial u_t}) + \ldots + \gamma^{(l)}(H - h\frac{\partial L}{\partial u_t})\right] \doteq 0. \quad (36)$$

Since $\gamma(t)$ is arbitrary we will get:

$$\begin{aligned}
&\gamma(t): && \iint dxdy\, D_t\left(A - a\frac{\partial L}{\partial u_t}\right) \doteq 0\\
&\gamma'(t): && \iint dxdy\left\{A - a\frac{\partial L}{\partial u_t} + D_t\left(B - b\frac{\partial L}{\partial u_t}\right)\right\} \doteq 0\\
&\gamma''(t): && \iint dxdy\left\{B - b\frac{\partial L}{\partial u_t} + D_t\left(C - c\frac{\partial L}{\partial u_t}\right)\right\} \doteq 0\\
&\quad\cdots\cdots\\
&\gamma^{(l+1)}(t): && \iint dxdy\left\{H - h\frac{\partial L}{\partial u_t}\right\} \doteq 0.
\end{aligned} \tag{37}$$

The system (37) can be written in the form:

$$\iint dxdy\left\{A - a\frac{\partial L}{\partial u_t}\right\} \doteq \iint dxdy\left\{B - b\frac{\partial L}{\partial u_t}\right\} \doteq \ldots \doteq \iint dxdy\left\{H - h\frac{\partial L}{\partial u_t}\right\} \doteq 0. \tag{38}$$

Obviously equations (38) do not determine an infinite number of conservation laws. It is possible to prove that for point and contact transformations (when coefficients of the symmetry vector α depend on x^i, u, u_i only) all equations (38) can be satisfied only in the whole space. Therefore in this case, only trivial conservation laws (with zero characteristics [2]) can be obtained.

4 Infinite symmetry algebra with arbitrary function of a spatial variable

Let α be a Noether symmetry of the form:

$$\alpha = ap(x) + bp'(x) + cp''(x) + \ldots + hp^{(l)}(x), \tag{39}$$

and let

$$M_t = Ap(x) + Bp'(x) + Cp''(x) + \ldots + Hp^{(l)}(x). \tag{40}$$

Using equation (31) and requiring strict boundary conditions (32) (in addition to the Noether boundary condition (29)) we will get the Noether conservation law (34). For the case $L = L(x^i, u, u_t)$ this equation will take a form

$$D_t \iint dxdy\left[p\left(A - a\frac{\partial L}{\partial u_t}\right) + p'\left(B - b\frac{\partial L}{\partial u_t}\right) + \ldots + p^{(l)}\left(H - h\frac{\partial L}{\partial u_t}\right)\right] \doteq 0. \tag{41}$$

Integrating (41) by parts and choosing $p(x)$, $p'(x)$... vanishing on the boundary, we will obtain the following conservation law:

$$D_t \iint p(x)dxdyR_t \doteq 0 \tag{42}$$

with a conserved density

$$R_t = \left(A - a\frac{\partial L}{\partial u_t}\right) - D_x\left(B - b\frac{\partial L}{\partial u_t}\right) + \ldots + (-1)^l (D_x)^l \left(H - h\frac{\partial L}{\partial u_t}\right). \tag{43}$$

Thus, an infinite symmetry group with an arbitrary function of a spatial variable leads to just one conservation law (43).

5 Equation of nonstationary transonic gas

As an example of the system with one-dimensional infinite symmetry algebra, let us consider the equation of nonstationary transonic gas [7]

$$2u_{xt} + u_x u_{xx} - u_{yy} = 0 \tag{44}$$

with Lagrangian

$$L = -u_x u_t - \frac{u_x^3}{6} + \frac{u_y^2}{2}. \tag{45}$$

Lie point symmetry group for the equation (44) was studied in [8]. Infinite set of conservation laws for this equation was given in [5]. We will consider an infinite subgroup of the symmetry group with an arbitrary function $\gamma(t)$:

$$X = \gamma(t)\frac{\partial}{\partial x} + (2\gamma'(t)x + 2\gamma''(t)y^2)\frac{\partial}{\partial u}, \tag{46}$$

or in canonical form

$$X_\alpha = \alpha\frac{\partial}{\partial u} + (D_i\alpha)\frac{\partial}{\partial u_i} + \sum_{i\le j}(D_i D_j \alpha)\frac{\partial}{\partial u_{ij}} + \cdots$$
$$\alpha = 2\gamma' x + 2\gamma'' y^2 - \gamma u_x. \tag{47}$$

Symmetry α is a divergence transformation, $X_\alpha L = D_i M_i$, with

$$M_x = -\gamma L - 2xu\gamma'' - 2y^2 u\gamma''', \quad M_y = 4yu\gamma'', \quad M_t = -2u\gamma'. \tag{48}$$

Demanding Noether boundary conditions (29), and strict boundary conditions (33):

$$\alpha(u_t + u_x^2/2) \underset{x\to\partial D}{\to} 0, \quad \alpha u_y \underset{y\to\partial D}{\to} 0 \tag{49}$$

we will get the following cases:

1) $\gamma(t)$ *is arbitrary.*

$$u, u_i \underset{x_i \to \partial D}{\to} 0, \quad xu \underset{x \to \partial D}{\to} 0, \quad yu \underset{y \to \partial D}{\to} 0, \quad xu_t, xu_x{}^2 \underset{x \to \partial D}{\to} 0, \quad y^2 u_y \underset{y \to \partial D}{\to} 0. \tag{50}$$

In this case no nontrivial local conservation laws are associated with the Noether transformation (47).

2) $\gamma'(t) = 0, \ \gamma(t) = c = const., \ u_i \underset{x_i \to \partial D}{\to} 0$

According to (36) we will get the following conservation law:

$$\iint u_x{}^2 dxdy \doteq const. \tag{51}$$

As we can see strict boundary conditions (50) determined nontrivial conservation laws.

6 Conclusion

We have shown that infinite symmetries with arbitrary functions of one of independent variables do not lead to an infinite number of conservation laws. Noether and strict boundary conditions determine the existence of associated nontrivial conservation laws.

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NORMAL FORMS, GEOMETRY, AND DYNAMICS OF ATOMIC AND MOLECULAR SYSTEMS WITH SYMMETRY

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We show how general ideas of Louis Michel on the application of topology, group theory, and commutative algebra in the qualitative analysis of symmetric dynamical systems can be concretized for finite-dimensional Hamiltonian dynamical systems with symmetries, notably atoms and molecules. This talk is a contribution to the special session in the memory of *Louis Michel*.

1 Introduction

In his work Louis Michel was always determined to extract as much information as possible from the symmetry and topology of the physical systems he studied; he used to call his approach as a program of *Symmetry and Topology*. His collaboration with Boris Zhilinskiĭ on the application of this general program to atomic and molecular systems resulted in a development of the method of qualitative analysis, a set of concrete mathematical tools, based on group theory, basic topology, Morse theory, commutative algebra, and invariant theory. We show on a number of concrete examples of real molecular and atomic systems, which represent the class of finite dimensional Hamiltonian dynamical systems with symmetries, how this method is implemented. A general review of the qualitative analysis can be found in Chapters I-III of [I]; our dynamical analysis, in particular the use of invariants for reduction (and singular reduction) of symmetries was greatly influenced by Richard Cushman.[II] We restricted the rest of the bibliography to our papers whose results were used in the talk; these papers should be looked up for the references to the related work by other authors.

2 Tools for qualitative and dynamical analysis

We give a brief summary of the stages of the qualitative and dynamical analysis of atomic and molecular systems.[I]

2.1 Classic qualitative approach developed by Michel and Zhilinskiĭ

We formulate the problem and study the symmetry group G of our system, typically a finite group which includes geometric and time-reversal operations. In certain cases we consider (construct) a classical analogue of a given quantum system. Otherwise, when the classical initial system is known *a priori*, we propose the *dynamical symmetry*, typically an approximate continuous S_1 symmetry, and the way to reduce this classical system. If the strict symmetry of the system G has continuous elements, we consider reduction of such symmetries as well. In all cases we end up studying a reduced classical Hamiltonian system defined, normally, over a compact reduced phase space P. We further study the topology of P, in particular we look for the Betty numbers of P.

We describe the action of the initial symmetry group G on the reduced phase space P. We use polynomial invariants of the dynamical symmetry as coordinates on P. We further symmetrize these invariants with respect to G and use them to describe the action of G on P. We construct the *orbit space* $\mathcal{O}$ of this action and consider the stratification of P. We find fixed points (*critical orbits*) and invariant subspaces of the group action.

We further study the commutative algebra of our invariants and the structure of the ring of all polynomial invariants. We find the *integrity basis*, which is a more sophisticated variety of the Gröbner basis and which allows to represent unambiguously any polynomial in this ring and therefore any analytic function on P in terms of invariants. Explicit construction of the integrity basis is greatly facilitated by the *Molien generating function* which we compute for the given group action on invariant polynomials.

We assume that our reduced Hamiltonian is in the class of simplest Morse functions on P. Such functions have the minimum number of stationary points compatible with topology and symmetry. These points are *relative equilibria* of the system. Points in the critical orbits of the group action are necessarily stationary, points in the same orbit are equivalent. Further restrictions on the stability of stationary points are imposed by their local symmetry (stabilizer). Taking all this into account we place stationary points on the critical orbits and check if the Morse inequalities and Euler equation for P are satisfied. Normally this singles out a few possible minimum sets of stationary points; it may occur, however, that additional stationary points must be placed on non-critical orbits in order to satisfy Morse theory. These latter points are also relative equilibria but their exact position on P depends on the concrete Hamiltonian.

We construct parametric families of simplest polynomial Morse Hamil-

tonians $\mathcal{H}$ from the integrity basis invariants. Such Hamiltonians are of the lowest degree which is sufficient to represent correctly the simplest G-invariant Morse function on P; they typically have a small number of parameters. We study $\mathcal{H}$ as a function on the orbit space $\mathcal{O}$. We classify constant level sets $\{\mathcal{H} = h\}$ of different $\mathcal{H}$ on $\mathcal{O}$. Geometrically we look for intersections of the surface $\mathcal{H} = \text{const}$ and the orbifold $\mathcal{O}$, both defined in invariant polynomial coordinates[a]. Relative equilibria (stationary points) of different kinds correspond to exceptional sections $\{\mathcal{H} = h\} \cap \mathcal{O}$ We find the energies of relative equilibria, i.e., the values h of $\mathcal{H}$ at such sections. Our qualitative analysis reaches its destination: each $\mathcal{H}$ is represented entirely by the family of its level sets on $\mathcal{O}$. We also consider bifurcations which lead to qualitatively different families of constant level sets and which occur when parameters of $\mathcal{H}$ are made to vary.

We transfer our results to quantum mechanics. We use energies of relative equilibria to define energy limits for the multiplets of quantum states. We predict localization patterns near stable relative equilibria, and corresponding regular sequences of levels. Finally we study quantum analogues of $\mathcal{H}$.

2.2 Analytical and dynamical analysis

The two important complements to the general scheme of Michel and Zhilinskií are based on the normalization (analytical aspect) and the Poisson structure of the reduced system (dynamical aspect). We normalize the initial classical Hamiltonian H explicitly in order to reduce the dynamical symmetry, the truncated normal form $H_{\text{n.f.}}$ is expressed in terms of dynamical invariants using integrity basis. The invariant which represents the approximate dynamical integral of motion is replaced by its value and becomes a parameter. This makes $H_{\text{n.f.}}$ a *reduced Hamiltonian* H_{red} on the reduced phase space P. We can also consider H_{red} as a function on the orbit space $\mathcal{O}$.

We study relative equilibria of H_{red} analytically, in particular we verify whether H_{red} is of the simplest Morse type. We consider H_{red} as a parametric family and look for bifurcations of relative equilibria, or, more generally, of constant level sets of H_{red} on $\mathcal{O}$. We reconstruct relative equilibria as periodic orbits of the original classical mechanical system. Furthermore, we reconstruct the entire geometry of (the integrable approximation to) the original system, i.e., we reconstruct all dynamically invariant subspaces of this system (invariant KAM tori) and describe how these subspaces fit together and foliate the constant energy level set of the initial phase space. We now can do full EBK quantization.

[a]This method of sections is also widely used by R. Cushman.

We equip the commutative algebra of the generators of the ring of invariant polynomials with a Poisson structure by appropriately restricting the Poisson brackets of generators. The latter become dynamical variables of the reduced system and we write Euler–Poisson equations of motion on the reduced phase space P. Though not general, this simple method works in all cases we discuss. We can further restrict these equations to invariant subspaces of P.

3 Hénon-Heiles system: an illustration

We follow the scheme of the qualitative analysis on the example of the well known model system, the three-fold symmetric perturbation of the 1:1 harmonic oscillator. The initial Hénon-Heiles Hamiltonian

$$H = H_0 + \epsilon V(q) = \frac{1}{2}(p_x^2 + p_x^2) + \frac{1}{2}(q_y^2 + q_y^2) + \epsilon\left(\frac{1}{3}q_x^3 - q_x q_y^2\right) \qquad (1)$$

is a function on R_4 with standard symplectic structure $q_x \wedge p_x + q_y \wedge p_y$. The spatial symmetry group of (1) is a dihedral group D_3, the full symmetry group is $D_3 \times \mathcal{T}$ where $\mathcal{T}$ is a Z_2 symmetry of the kind $(q,p) \to (q,-p)$ often called *time reversal* or *momentum reversal*.

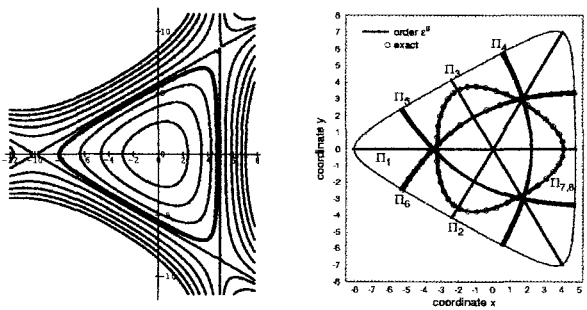

Figure 1. Hénon-Heiles potential $V(q)$ calculated for $\epsilon = 0.1$ and $E/E_{\text{saddle}} = 0.2, 0.45, 0.7, \mathbf{0.9}, 1, 1.2, \ldots$ (left); Relative equilibria (nonlinear normal modes) of the Hénon-Heiles system reconstructed from the ϵ^8 normal form at the energy $E/E_{\text{saddle}} = 0.9$ (right).

3.1 Dynamical symmetry, invariants, reduced phase space

The dynamical S_1 symmetry is generated by the flow of the unperturbed 1:1 oscillator. Using variables $z = q + ip$ we can write the four quadratic invariants

of this symmetry as follows

$$H_0 = 2J = \frac{1}{2}(z_x\bar{z}_x + z_y\bar{z}_y), \quad \mathbf{J} = \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} z_x\bar{z}_y + z_y\bar{z}_x \\ iz_x\bar{z}_y - iz_y\bar{z}_x \\ z_x\bar{z}_x - z_y\bar{z}_y \end{pmatrix}. \quad (2)$$

The Poisson algebra generated by the components of the 3-vector $\mathbf{J}$ is the standard so(3) with bracket $\{J_a, J_b\} = \epsilon_{abc}J_c$ and Casimir $J = |\mathbf{J}|$ (or H_0). For each constant value of the dynamical integral J we construct the reduced phase space P^J as a 2-sphere S_2^J embedded in the ambient space R_3 with coordinates (J_1, J_2, J_3). The defining equation of P^J is, of course,

$$J^2 = J_1^2 + J_2^2 + J_3^2 = \text{const}, \quad (3)$$

which is also the only algebraic relation between (J_1, J_2, J_3).

3.2 Group action, critical orbits, and stationary points

The action of the symmetry group $D_3 \times \mathcal{T}$ on S_2^J is equivalent to the action of the point group D_{3h} of transformations of the R_3 space[b] which acts naturally on $S_2^J \subset R_3$. Since J_2 is invariant with respect to any rotation of the initial coordinates (x, y), it is convenient to choose J_2 along the vertical axis. Then time reversal $\mathcal{T}$ acts as the horizontal reflection plane $(J_1, J_2, J_3) \to (J_1, -J_2, J_3)$. The action of $D_3 \times \mathcal{T}$ has 8 fixed points which form three critical orbits characterized in table 1.

Table 1. Critical orbits of the $D_3 \times \mathcal{T} \sim D_{3h}$ action on the reduced phase space S_2^J of the Hénon-Heiles system. The $C_3 \wedge \mathcal{T}_2$ subgroup of $D_3 \times \mathcal{T}$ is generated by C_3 and $\mathcal{T}_2 = C_2 \circ \mathcal{T}$; the groups $C_3 \wedge \mathcal{T}_2$, D_3, and C_{3v} are isomorphic as abstract groups. "Historic" labels Π_k of the were introduced for the nonlinear normal modes by Montaldi, Roberts and Stewart and used later in [2]. Note that $(J_1, J_2, J_3) = J(\sin\theta\sin\varphi, \cos\theta, \sin\theta\cos\varphi)$.

orbit	stabilizer	ξ/J^3	μ/J^2	θ	φ
$\Pi_{7,8}$	$C_3 \wedge \mathcal{T}_2$	0	1	0	–
$\Pi_{4,5,6}$	$C_2 \times \mathcal{T}$	$-1/2$	0	$\pi/2$	$0, \pm 2\pi/3$
$\Pi_{1,2,3}$	$C_2' \times \mathcal{T}$	$1/2$	0	$\pi/2$	$\pi, \pm\pi/3$

The simplest D_{3h} invariant Morse function $\mathcal{H}$ on S_2 has eight stationary points which are situated on the critical orbits described in table 1. These

[b] We use Schönflis notation for point groups, see M. Hamermesh, *Group theory and its application to physical problems* (Addison-Wesley, Reading, 1964).

points should satisfy the Euler relation for the sphere S_2

$$c_0 - c_1 + c_2 = 2,$$

where c_0, c_1, and c_2 are the numbers of maxima, saddle points, and minima. Furthermore, since the two equivalent $\Pi_{7,8}$ points on S_2 have the stabilizer C_3, the three-fold rotation symmetry, these points should be elliptic (stable). Of the two three-point orbits with stabilizers C_2 and C_2', one should accommodate elliptic points and the other—hyperbolic (unstable). One possible simplest Morse function is drawn in fig. 2, left. It has two equivalent maxima at $\Pi_{7,8}$, three equivalent minima at $\Pi_{1,2,3}$ and three equivalent saddle points at $\Pi_{4,5,6}$.

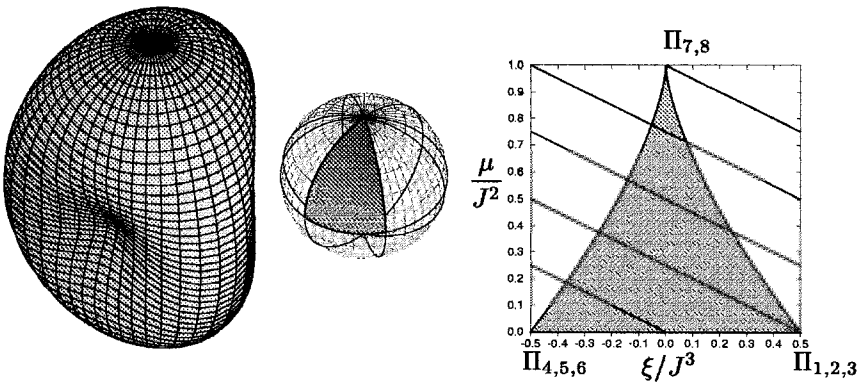

Figure 2. Relative equilibria of the Hénon-Heiles system as stationary points of the reduced Hamiltonian $H^J_{\text{n.f.}}$ on the reduced phase space S^J_2. On the left we show $H^J_{\text{n.f.}}$ as a function on S^J_2. The shaded area on the right and central panel represents the orbit space (orbifold) $\mathcal{O}$ of the $D_3 \times \mathcal{T}$ action on S_2; straight lines in the right panel are constant level sets of the simplest $D_3 \times \mathcal{T}$-invariant Morse Hamiltonian $\mathcal{H} = \mu + \epsilon\xi$.

3.3 Integrity basis

The generating function for the polynomials in four initial phase space variables $(z_x, z_y, \bar{z}_x, \bar{z}_y)$ invariant with respect to the S_1 symmetry of the harmonic oscillator

$$g(\lambda) = (1 + \lambda^2)/(1 - \lambda^2)^3, \tag{4}$$

where the formal variable λ represents any $(z, \bar{z})$, is computed directly from the Molien theorem.[11] The ring $\mathcal{R}$ of invariant polynomials generated by (J_1, J_2, J_3) is not free because of the algebraic relation (3). We can decompose this ring as $\mathcal{P}(J, J_1, J_2) \cdot \{1, J_3\}$ meaning that any member in $\mathcal{R}$ can be

expressed using terms $J^a J_1^b J_2^c$ and $J^a J_1^b J_2^c J_3$ with (a, b, c) arbitrary nonzero integers. For instance, any power of J_3 can be represented this way using (3). The structure of $\mathcal{R}$ is confirmed by the Molien function (4) which tells that there are three *principal* or *denominator invariants* in $\mathcal{P}$ and one nontrivial *auxiliary* or *numerator invariant* in $\{\}$, all of degree 2 in $(z, \bar{z})$. This kind of information is invaluable in higher dimensional situations.

The group $D_3 \times \mathcal{T}$ acts on (J_3, J_1, J_2) in the same way as D_{3h} acts on (x, y, z), the polynomials (J_3, J_1, J_2) span the $E \oplus A_2$ representation of this group. The Molien generating function for $D_3 \times \mathcal{T}$ invariant polynomials in (J_1, J_2, J_3)

$$g(E \oplus A_2 \to A_1; \lambda) = \frac{1}{(1-\lambda^2)(1-\lambda^3)}$$

can be obtained straightforwardly from the action of the finite group $D_3 \times \mathcal{T}$ on (J_1, J_2, J_3). We conclude that the ring of all polynomial invariants of the combined action of $D_3 \times \mathcal{T}$ and oscillator symmetry S_1 is freely generated by (n, μ, ξ) where $n = 2J$ is the main oscillator invariant [see (2)] and μ and ξ are polynomials in (J_1, J_2, J_3) of degree 2 and 3 respectively. The generators can be chosen explicitly as follows

$$n = 2J, \quad \mu = J_2^2, \quad \xi = \frac{1}{2} J_3 (3J_1^2 - J_3^2). \tag{5}$$

In simple terms this means that the normalized Hénon-Heiles Hamiltonian is a function $H_{\text{n.f.}}(n, \mu, \xi)$ with n later relegated as a parameter. To our knowledge, this result of the application of elementary invariant theory has been not appreciated by the numerous studies of the Hénon-Heiles system.

3.4 Orbit space of the $D_3 \times \mathcal{T}$ action on S_2

As shown in fig. 2, right, the orbit space $\mathcal{O}$ of the $D_3 \times \mathcal{T}$ action on S_2 can be defined as an algebraic variety in the 2-plane with coordinates (μ, ξ)

$$0 \le \frac{\mu}{J^2} \le 1 - t^2, \quad \frac{|\xi|}{J^3} \le \frac{1}{2} t^3, \quad t \in [0, 1].$$

Knowing the D_{3h} action on S_2 (fig. 2, centre) we can see that $\mathcal{O}$ is the image of the triangular petal on S_2 cut out by three symmetry planes, two vertical planes intersecting at the angle $\pi/3$ and the horizontal plane. Those who prefer using algebra will consider

$$\det \left[\frac{\partial(\mu, \xi, J)}{\partial(J_1, J_2, J_3)} \right] = -6 J_1 J_2 (3J_3^2 - J_1^2) = 0$$

and observe that the borders and singular points of $\mathcal{O}$ (one- and zero-dimensional strata on S_2) correspond to simple and double zeroes of this Jacobian.

3.5 Simplest polynomial Morse Hamiltonian

Since the invariant μ has high axial symmetry S_1, the simplest polynomial Hamiltonian $\mathcal{H}$ should include the higher degree term ξ in order to have correct symmetry $D_3 \times \mathcal{T}$. Generally, at least when J is low, we expect the contribution by ξ which has higher degree in (J_1, J_2, J_3) to be less important than that of μ. The function $\mathcal{H} = a\mu + b\xi$ with $a \gg b$ is illustrated in fig. 2, left. This function is of the simplest Morse type. The family of constant level sets of $\mathcal{H}$ on the orbifold $\mathcal{O}$ has three exceptional (critical) levels which pass at $\Pi_{1,2,3}$, $\Pi_{4,5,6}$, and $\Pi_{7,8}$ (see fig. 2). The extremal levels correspond necessarily to stable relative equilibria, the critical level at the intermediate energy $\mathcal{H}_{\Pi_{4,5,6}}$ contains unstable relative equilibria and their stable/unstable manifold (separatrix). Stability of the stationary points Π of $\mathcal{H}$ is given by the frequency ω of small oscillations about them, i.e, by the coefficient in the linearization of $\mathcal{H}$ near Π. Frequencies $\omega_{\Pi_{7,8}}$ and $\omega_{\Pi_{1,2,3}}$ are defined primarily by μ and ξ contributions respectively. If $\mathcal{H} = \mu + \epsilon\xi$, $\epsilon \ll 1$, and n is small, then $\omega_{\Pi_{7,8}} \gg \omega_{\Pi_{1,2,3}}$, i.e., the $\Pi_{7,8}$ maximum is much "deeper" than $\Pi_{1,2,3}$, cf. fig. 2.

3.6 Qualitative reconstruction of relative equilibria

Stationary points of $\mathcal{H}$ are relative equilibria of the system. They correspond to special periodic orbits of the system (see fig. 1, right), which we also call nonlinear normal modes (nnm). Stability and symmetry properties of these periodic orbits is defined by stability ω and local symmetry (stabilizer) of relative equilibria. The two elliptic nnm's $\Pi_{7,8}$ are not $\mathcal{T}$-invariant and form loops in the configuration plane with coordinates (x, y). These nnm's share the same D_3 symmetric image in this plane and they run in opposite directions so that the time reversal $\mathcal{T}$ sends $\Pi_7 \leftrightarrow \Pi_8$. Periodic orbits with stabilizer $C_2 \times \mathcal{T}$ or $C_2' \times \mathcal{T}$ are $\mathcal{T}$-invariant and their (x, y) image is a line with ends connecting to the border of the allowed coordinate domain (where kinetic energy is zero) at a right angle. Since the (x, y) image is also C_2-invariant, three such nnm's lie on the C_2 axes, and three others cross these axes at a right angle. We should mention that the above qualitative description of relative equilibria was first obtained on the basis of a different approach by Montaldi, Roberts, and Stewart in 1988.

Qualitative reconstruction of invariant manifolds of (the integrable approximation to) the Hénon-Heiles system starts with the level sets $\{\mathcal{H} = h\} \cap \mathcal{O}$ (fig. 2) which should be first brought up to the reduce phase space S_2. This can be quickly done using a bit of geometric intuition. Regular sets Γ with energies $H_{\Pi_{4,5,6}} < h_\Gamma < H_{\Pi_{7,8}}$ map to two equivalent circular periodic orbits on S_2 (one goes around Π_8 and another—around Π_7), while sets Γ' with energies $H_{\Pi_{1,2,3}} < h_{\Gamma'} < H_{\Pi_{4,5,6}}$ become three equivalent periodic orbits on S_2 (each going around one of the points $\Pi_{1,2,3}$). To understand how these circular orbits fit together on S_2 imagine sections of the surface in fig. 2 by spheres of different radii h. (Pictures of this kind should be familiar to the fans of the Hénon-Heiles system or of the so-called rotational energy surfaces.) Going all the way to the initial phase space R_4 is easy once we remember that each point on the reduced phase space S_2 lifts to a circle in R_4. Therefore, Γ and Γ' lift to two and three equivalent regular 2-tori respectively.

3.7 Qualitative structure of quantum energy level spectrum

Quantized 2-oscillator integral n in (5) should equal $N+1$ where $1 = \frac{1}{2} + \frac{1}{2}$ is the Maslov correction for each oscillator degree of freedom and N takes integer values 0, 1, etc. Quantum number J equals $N/2$, the classical amplitude $j = \frac{1}{2}n$ satisfies quantum formula $\sqrt{J(J+1)}$ in the classical limit of $J \gg 1$ where $J(J+1) \approx (J+\frac{1}{2})^2$. The multiplet of quantum eigenstates with given N is called *polyad* and N is the polyad quantum number. Since our reduced phase space S_2 is compact, the number of states in each polyad is finite; for S_2^N this number equals $N+1 = 2J+1$. Quantum spectrum of the N-polyad is bracketed by the absolute maximum and minimum energy of the reduced Hamiltonian H_{red} at fixed n. These extremal energies correspond, of course, to certain stable relative equilibria (RE); in the case of the simplest Hamiltonian $\mathcal{H}$ they are $\mathcal{H}_{\Pi_{7,8}}$ and $\mathcal{H}_{\Pi_{1,2,3}}$.

Quantum wavefunctions localize near (sufficiently) stable RE. Localized wavefunctions exhibit characteristic one-dimensional nodal patterns along the corresponding periodic orbits in fig. 1, right. The number of nodes equals N. Near the energy of k equivalent RE we expect to find a quasidegenerate group of k levels split by tunneling. In the case of the simplest $D_3 \times \mathcal{T}$ invariant Hamiltonian $\mathcal{H}$ we should find doublets and triplets that form quasi-equidistant one-dimensional oscillator sequences beginning at $\mathcal{H}_{\Pi_{7,8}}$ and $\mathcal{H}_{\Pi_{1,2,3}}$ respectively. The initial spacing between the neighbors in these sequences is defined by the stabilities $\omega_{\Pi_{7,8}}$ and $\omega_{\Pi_{1,2,3}}$. The energy $\mathcal{H}_{\Pi_{4,5,6}}$ of the unstable RE separates the two sequences. Since we argued that $\omega_{\Pi_{7,8}} \gg \omega_{\Pi_{1,2,3}}$, we should expect the doublets to occupy a larger energy

domain and to be more numerous and better pronounced (when J is low).

3.8 Analysis based on the normal form

We normalize (1) using the standard Lie transformation method. To order ϵ^4 (second order transformation) the normal form of (1) is[c]

$$H_{\text{n.f.}} = n - \epsilon^2 \left(\frac{5}{12} n^2 - \frac{7}{3} \mu \right) - \epsilon^4 \left(\frac{67}{432} n^3 + \frac{7}{36} \mu n - \frac{56}{9} \xi \right) + \dots , \qquad (6)$$

coefficients in the higher orders are listed below.

order	1	μn^{-2}	ξn^{-3}	$\mu^2 n^{-4}$	$\mu\xi n^{-5}$
$\epsilon^6 n^4$	$-\frac{42229}{155520}$	$-\frac{76447}{6480}$	$\frac{2093}{135}$	$\frac{115171}{1944}$	
$\epsilon^8 n^5$	$-\frac{15624833}{18662400}$	$-\frac{11656729}{2332800}$	$\frac{353843}{8100}$	$\frac{2217943}{233280}$	$\frac{6701639}{4050}$

The energies of relative equilibria are the values of this $H_{\text{n.f.}}$ at the corresponding values of (μ, ξ) in table 1,

$$H_{\Pi_{7,8}} = n + \frac{1}{6}\epsilon^2 n^2 - \frac{11}{54}\epsilon^4 n^3 + \frac{1171}{2430}\epsilon^6 n^4 - \frac{217567}{145800}\epsilon^8 n^5 \qquad (7a)$$

$$H_{\Pi_{1,2,3}} = n - \frac{5}{12}\epsilon^2 n^2 - \frac{235}{432}\epsilon^4 n^3 - \frac{38585}{31104}\epsilon^6 n^4 - \frac{2663129}{746496}\epsilon^8 n^5 \qquad (7b)$$

$$H_{\Pi_{4,5,6}} = n - \frac{5}{12}\epsilon^2 n^2 + \frac{101}{432}\epsilon^4 n^3 + \frac{108467}{155520}\epsilon^6 n^4 + \frac{35328559}{18662400}\epsilon^8 n^5 \qquad (7c)$$

Each relative equilibrium is a periodic orbit in the initial phase space R_4. It becomes an orbit of the dynamical S_1 symmetry in the transformed phase space $\tilde{R}_4$ after normalization $R_4 \to \tilde{R}_4$. The integral n is the action integral along this orbit,

$$n(h) = \frac{1}{2\pi} \oint (p_x dq_x + p_y dq_y), \qquad (8)$$

taken at fixed energy $H = h$. If we inverse the formal series (7) we obtain, therefore, the value of action (8) in the original space R_4 as a function of energy h for each nonlinear normal mode (each RE). The period of the orbits is then found as $T = 2\pi(dn/dh)$. Results are compared in fig. 3 with data obtained by numerical integration.

We can reconstruct explicitly all dynamically invariant subspaces of (the integrable approximation to) the Hénon-Heiles system. To this end we lift points on the base space S_2^J back to the total space $\tilde{R}_4$ and then use the

[c]Note, that we may differ from other authors in the choice of $\mathbf{J}$, or of its particular components. Thus Churchill and Rod obtain the same normal form (6) in terms of the components of $\mathbf{W} = 4\mathbf{J}$. They do not use $D_3 \times \mathcal{T}$-invariant polynomials μ and ξ.

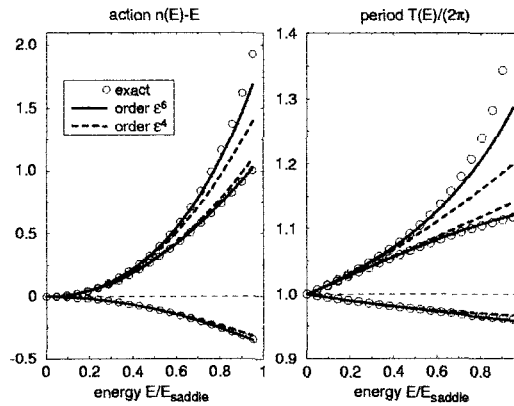

Figure 3. Action $n(E)$ and period $T(E)$ of relative equilibria (right). Solid and dashed lines show normal form approximations of order ε^6 and ε^4, circles represent exact numerical data.

formal Lie series for the inverse transformation $\tilde{R}_4 \to R_4$. From the geometric point of view, we reconstruct analytically the Hopf fibration[11] with the base S_2^J, fiber S_1, and total space S_3, which is the constant energy level set in R_4. In the simplest case of a relative equilibrium, we take just one point in S_2^J. Resulting periodic orbits in R_4 projected on the configuration plane (x, y) are shown in fig. 1. Even at high energies h they compare quite favorably to the orbits of the original non-integrable system provided that the $H_{\text{n.f.}}$ is taken to a sufficiently high order. In the general situation, we first use Euler equations of motion $\{H^J_{\text{n.f.}}(J_1, J_2, J_3), J_k\}$ in order to reconstruct the dynamics of the reduced system.

3.9 Quantized normal form

We can construct a quantum analogue of $H_{\text{n.f.}}$ in (6) if we re-express $H_{\text{n.f.}}$ as function of (J_1, J_2, J_3) using (5) and then replace (J_1, J_2, J_3) for quantum angular momentum operators. We can also return to the initial coordinates $(z, \bar{z})$ and then replace them with creation-annihilation oscillator operators[d] $(\sqrt{2}a, \sqrt{2}a^+)$. In both cases purists will accuse us of ambiguity because quantum elementary operators $(\hat{J}_1, \hat{J}_2, \hat{J}_3)$ or (a, a^+) do not commute and their correct ordering is lost in the process of normalization. Indeed, a true quantum analogue should be constructed using a parallel quantum normalization

[d]This is called the *Schwinger representation* of the angular momentum system.

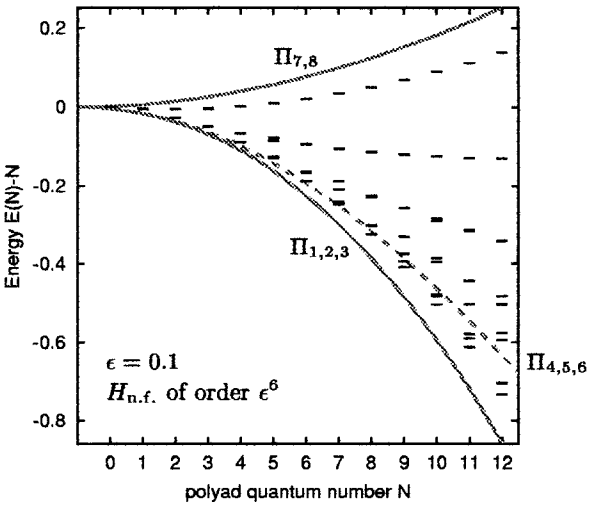

Figure 4. Energies of relative equilibria (solid and dashed lines) and quantum energy levels (bars) of the Hénon-Heiles system with $\epsilon = 0.1$ obtained using order ϵ^6 normal form $H_{\text{n.f.}}$ as functions of the polyad quantum number N, here the classical action n equals $N + 1$.

procedure known as contact or Van Vleck transformation. The general response is that our quantum $\widehat{H}_{\text{n.f.}}$ should work well in the classical limit where n is large *and* ϵ is small. In such limit, commutator corrections to the given principal order which come from higher orders of $H_{\text{n.f.}}$ can be comfortably neglected. This does not, of course, exhaust all areas of interest, but is sufficient for the illustration of our qualitative predictions.

Figure 4 shows the spectrum of $\widehat{H}_{\text{n.f.}}$ in the form of quantum-classical energy-momentum diagram. The bars which show quantum levels give positions of classical EBK 2-tori amended by tunneling. With this correction in mind, we can associate this figure with the image of the energy-momentum map in the usual sense of Cushman.[11] The larger upper part of this spectrum is occupied by doublets. The $N = 12$ polyad has three such doublets but their splitting is much too small to be reproduced in the figure. In addition, the states in the doublets realize representations $A_1 \oplus A_2$ or E of the D_3 group and in the latter case they are strictly degenerated due to symmetry. At bottom energies of large-N polyads in fig. 4 lies a triplet level. Quantum states in these triplets realize the representation $A \oplus E$, so all we can see is a doublet which is split considerably due to the low barrier energy $H_{\Pi_{4,5,6}} - H_{\Pi_{1,2,3}}$. We

conclude that our qualitative predictions are indisputably correct.

4 Real atomic and molecular systems

Our atomic examples use hydrogen atom perturbed by different configurations of external electric and magnetic fields. It can be easily generalized to other perturbed Kepler systems. Our analysis of the quadratic Zeeman effect serves to establish the relation of our approach and terminology to the numerous previous work by other authors which goes back to the pioneering study of Solov'iev in 1981. Our major contribution to the study of the parallel field system is the analysis of the so-called crossover (or collapse) phenomenon[5] which can be done in the elegant style of Michel and Zhilinskiĭ based almost entirely on the invariant theory and the orbit space analysis of simplest Hamiltonians. Our collaboration with Richard Cushman[7] on the hydrogen atom in crossed fields[6] resulted in uncovering monodromy of one of the most fundamental physical systems, in fact one of the few quantum systems where monodromy can be observed and studied experimentally.

We also consider molecular vibration and rotation-vibration systems[1,3,4]. The H_3^+ molecular ion[2] has the equilibrium configuration of an equilateral triangle. Near this equilibrium, it has three internal vibrational degrees of freedom, the "breathing" nondegenerate mode ν_1 and the doubly-degenerated "bending" mode ν_2. These modes transform according to the irreducible representations A_1 and E of the C_{3v} symmetry group of the molecule. The E-mode system at low energies is similar to the Hénon-Heiles oscillator; it has the same 8 relative equilibria. Quantum states localized near some of these modes are found in [2]. Tetrahedral molecules A_4 and AB_4 have a rich system of rotational and rotational-vibrational relative equilibria. We analyze them in detail and study the relation of the energy of relative equilibria and corresponding structure of quantum rotation-vibration energy levels to the parameters of the molecular potential.[9,10,11]

5 Concluding remarks

Our purpose was to demonstrate the influence of Louis Michel at all levels of our approach to the analysis of atomic and molecular systems. We tried to convey the taste of the intricate combination of ideas, methods, and tools, all cemented by the strong belief in the possibility and necessity to achieve an elegant and simple understanding of concrete physical problems on the basis of symmetry and topology. We shall remember Louis Michel for his great physical intuition, wide and solid mathematical background, and his courage

to apply constructively the most modern mathematical theories. We hope that our work can serve as a model for further cooperation of contemporary physics and mathematics. The work continues both in the development of theory and methods and in applications to more complex systems. These new developments could only be mentioned in the talk. In particular we should pay attention to the effort of understanding quantum and classical monodromy in real systems which are only approximately integrable, and in systems with more than two degrees of freedom. We should also continue to uncover the relation between monodromy and topological quantum numbers in mixed quantum-classical systems.[8]

Acknowledgments

This talk was invited by Boris Zhilinskií, who was a close colleague of Louis Michel and the driving force behind much of the work presented in the talk. Support by the EU project *Mechanics and Symmetry in Europe* (MASIE), contract HPRN-CT-2000-00113, is gratefully acknowledged. We like to thank Giuseppe Gaeta for his kind encouragement.

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HIGHER ORDER RESONANCE IN TWO DEGREE OF FREEDOM HAMILTONIAN SYSTEM

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This paper reviews higher order resonance in two degrees of freedom Hamiltonian systems. We consider a positive semi-definite Hamiltonian around the origin. Using normal form theory, we give an estimate of the size of the domain where interesting dynamics takes place, which is an improvement of the one previously known. Using a geometric numerical integration approach, we investigate this in the elastic pendulum to find additional evidence that our estimate is sharp. In an extreme case of higher order resonance, we show that phase interaction between the degrees of freedom occurs on a short time-scale, although there is no energy interchange.

1 Introduction

Studies on two degrees of freedom Hamiltonian system has a long history: dating back to Euler in 1772 with his description of three body problem. The presence of resonance in a Hamiltonian system of differential equations, strongly affects the dynamics. In this paper, we give a review of studies on higher order resonance in two degrees of freedom Hamiltonian system. An early discussion of this problem, using formal methods, is given by Kevorkian[6].

The analysis in this paper involves asymptotic analysis (perturbation method) and normalization. Using these techniques, we construct an approximation to the original system and then study the dynamics of the approximate system. We also use a numerical method to gain confirmation of our analytical result. The numerical method that we used is based heavily on a geometrical approach to preserve some of the geometric structure of the system.

We start this paper with a mathematical setting of two degrees of freedom Hamiltonian system in $\mathbb{R}^4$. We also give a brief idea of the normalization in this section. For introduction to Hamiltonian system, see Abraham and Marsden[1] or Arnol'd[2], for normalization see Churchil et.al.[4]. In Section 3 we study the resonance domain, which is the main focus of interest in higher order resonance. Using normal form, we show that we can improve the estimate of

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the size of this domain. We check this statement also numerically and it is presented in Section 4. The last section, we present part of an on going research on an extreme type of higher order resonance.

2 Two degrees of freedom Hamiltonian systems at higher order resonance

Consider a two degrees of freedom Hamiltonian system, defined in $\mathbb{R}^4$ with coordinate $(\boldsymbol{q},\boldsymbol{p}) = (q_1, q_2, p_1, p_2)$ and a symplectic form $\mathrm{d}\boldsymbol{q}\wedge\mathrm{d}\boldsymbol{p} = \mathrm{d}q_1\wedge\mathrm{d}p_1 + \mathrm{d}q_2\wedge\mathrm{d}p_2$, with Hamiltonian

$$H = \frac{1}{2}\omega_1({q_1}^2 + {p_1}^2) + \frac{1}{2}\omega_2({q_2}^2 + {p_2}^2) + H_3(\boldsymbol{q},\boldsymbol{p}) + H_4(\boldsymbol{q},\boldsymbol{p}) + \dots, \quad (1)$$

where H_k is a homogeneous polynomial of degree k and $\omega_1, \omega_2 > 0$. Let ε be a small parameter ($\varepsilon \ll 1$). We localize in the neighborhood of the origin by rescaling the variables by $q_1 = \varepsilon\overline{q_1}$, $q_2 = \varepsilon\overline{q_2}$, $p_1 = \varepsilon\overline{p_1}$, and $p_2 = \varepsilon\overline{p_2}$. Dividing the Hamiltonian (1) by ε^2, we arrive at the Hamiltonian

$$H = \frac{1}{2}\omega_1({q_1}^2 + {p_1}^2) + \frac{1}{2}\omega_2({q_2}^2 + {p_2}^2) + \varepsilon H_3(\boldsymbol{q},\boldsymbol{p}) + \varepsilon^2 H_4(\boldsymbol{q},\boldsymbol{p}) + \dots, \quad (2)$$

where we have dropped the bar.

The idea of Birkhoff normalization is to transform the Hamiltonian such that it depend only on the so-called *action variables*. In general, this is only possible in the *non-resonance* situation. We proceed by bringing the Hamiltonian (2) to the *Birkhoff-Gustavson* normal form. Consider the *resonance relation*: $n\omega_1 + m\omega_2 = 0$. Let (n, m) integer solution of the resonance relation such that $|n| + |m| = r$ while m and n are relatively prime. Introducing the action variables $\tau_j = ({q_j}^2 + {p_j}^2)/2$ and the angle variables $\varphi_j = \arctan(p_j/q_j)$, $j = 1, 2$, the Hamiltonian (2) is transformed to the normal form of the form

$$H = \omega_1\tau_1 + \omega_1\tau_1 + \mathcal{P}(\tau_1, \tau_2, \varepsilon) + \varepsilon^{m+n-2}\mathcal{R}(\tau_1, \tau_2, \varphi_1, \varphi_2), \quad (3)$$

where $\mathcal{P}$ is a polynomial of degree $[\![r/2]\!]$. Moreover, the function $\mathcal{R}$ depend only on the *resonance combination angle*: $n\varphi_1 + m\varphi_2$ (instead of an individual angle).

3 The resonance domain

In this paper we are interested on the *higher order resonance* cases: $r \geq 5$. For the lower order resonance ($r = 3$ and $r = 4$), see for instance Nayfeh

and Mook[7], or van der Burgh[10]. Sanders[8] is one of the first who described the dynamics of (2) at high order resonance. He found that the phase-space is foliated by invariant tori parameterized by taking the actions τ_1 and τ_2 to be constant. Using the KAM theorem, most of these tori persist under a Hamiltonian perturbation. There exists also the so-called *resonant manifold.* A neighborhood of this manifold, called the *resonance domain*, is the location where interesting dynamics is found. At each energy level, the normal form produces at least one elliptic periodic solution and one saddle type periodic solution; they lie in the resonance manifold which is embedded in the energy manifold.

Writing $\varphi = n\varphi_1 + m\varphi_2$ as the resonance combination angle, the equations of motion derived from the normal form (3) is

$$\begin{aligned} \dot{\tau}_j &= \varepsilon^{m+n-4} \frac{\partial \mathcal{R}}{\partial \varphi_j}, \qquad j = 1,2 \\ \dot{\varphi} &= n\frac{\partial P_2}{\partial \tau_1} + m\frac{\partial P_2}{\partial \tau_2} + O(\varepsilon), \end{aligned} \tag{4}$$

where we have re-scaled time by $\varepsilon^2 t$, and P_2 is the quadratic part of $\mathcal{P}$. Consider a system of two linear equation

$$\begin{cases} m\tau_1 + n\tau_2 = E_0 \\ n\frac{\partial P_2}{\partial \tau_1} + m\frac{\partial P_2}{\partial \tau_2} = 0, \end{cases} \tag{5}$$

for some $E_0 \in \mathbb{R}^+$. Note that the first equation in (5) represents nothing but the approximate energy manifold $H_2 = E_0$. If

$$\Delta = \begin{vmatrix} m & n \\ n\frac{\partial^2 P_2}{\partial \tau_1^2} & m\frac{\partial^2 P_2}{\partial \tau_2^2} \end{vmatrix} \neq 0,$$

then the location of the resonance domain can be approximated by the solution of (5). Moreover, transfer of energy (or interaction) between the degrees of freedom occurs in the neighborhood of that solution. Sanders[8] also gave an estimate for the size, d_ε, of the resonance domain, which is of $O(\varepsilon^{-(m+n-4)/6})$, and the time-scale of interaction is $\varepsilon^{-(m+n)/2}$.

In van den Broek[9], numerical evidence has been given that the size is actually smaller than the above estimation, so there is room for improvement of the estimate of d_ε. In Tuwankotta and Verhulst[12], we use a geometric approach to give an estimate for d_ε. The idea is to use the normal form theory to construct an approximation of Poincaré section of the flow of (2). Suppose we construct the section in the q_1-p_1-direction. Each of the periodic solutions (inside the resonance domain) mentioned above will appear as an n-periodic point of the map. Obviously, there will be $2n$ of such points in the section (since we have two periodic orbits). The saddle type point will be

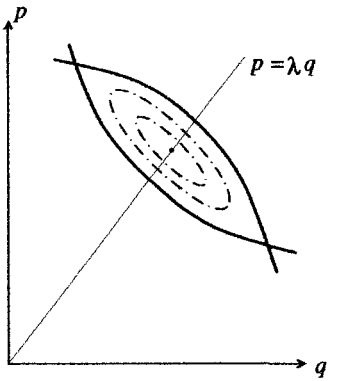

Figure 1. An illustration of calculation of the estimate for the d_ε.

connected to its neighboring saddle point with a heteroclinic cycle. We draw a line connecting the origin and one of the elliptic points. This line will intersect the heteroclinic cycle. The estimate for d_ε in Tuwankotta and Verhulst[12] is achieved by estimating the distance between the two intersection points. We summarize in the following lemma.

Lemma 3.1 *In two degrees of freedom Hamiltonian systems at higher order resonance $m : n$ with m and n natural numbers satisfying $m + n \geq 5$, the size d_ε of the resonance domain is*

$$d_\varepsilon = O(\varepsilon^{\frac{m+n-4}{2}}), \tag{6}$$

with a time-scale of interaction $O(\varepsilon^{-(m+n)/2})$.

The presence of an appropriate discrete symmetry completely changes the hierarchy of resonances in the system. As demonstrated in Tuwankotta and Verhulst[12], the 1 : 2-resonance for instance, has to be viewed as a 2 : 4-resonance in the presence of mirror symmetry in the second degree of freedom. As a consequence, it becomes a higher order resonance and thus the lemma above holds. The number of periodic solutions in the resonance manifold embedded in the energy manifold is then doubled.

4 Geometric Integration based on a splitting method

As mentioned previously, the first indication that the estimate of d_ε can be improved, is found numerically. The next thing for us to do is then to check the

estimate (6) numerically. For this purpose, we choose the Elastic Pendulum, which is a classical mechanical problem with discrete symmetry. This system serves as a model of a lot of applications, see the references in Tuwankotta and Verhulst[12].

The elastic pendulum is a mathematical pendulum in which the rod is replaced by a linear spring. The Hamiltonian of the elastic pendulum is

$$H = \frac{1}{2ml^2}\left(p_z{}^2 + \frac{p_\varphi{}^2}{(z+1)^2}\right) + \frac{sl^2}{2}\left(z + \frac{l-l_o}{l}\right)^2 - mgl(z+1)\cos(\varphi), \qquad (7)$$

where φ is the deviation from the vertical position, z is the radial oscillations, s is the spring constant, m is the mass of the pendulum, and l is the after load length of the pendulum. See Nayfeh[7],van der Burgh[10],Tuwankotta and Verhulst[12],Tuwankotta and Quispel[11] for details.

Due to the size of the domain which is relatively small, and the time-scale of interaction which is relatively long, it is not easy to get the numerical confirmation of the estimate (6). We need a method of time integration that is accurate enough after relatively long integration time as well as being fast in the real time computation.

By mean of an example, Tuwankotta and Quispel [11] demonstrated how the *Baker-Campbell-Hausdorf* (BCH) formula can be used to construct an approximation of the flow of (7). The idea is to split the Hamiltonian (7) into parts which are individually integrable. By composing the exact flow of each part, and using the BCH formula, the numerical integration scheme is then constructed. Using this method, an independent confirmation that our estimate is sharp, is achieved. See Tuwankotta and Quispel[11] for details, Table 1 for numerical result and also Figure (2).

Resonance	Resonant part	Analytic $\log_\varepsilon(d_\varepsilon)$	Numerical $\log_\varepsilon(d_\varepsilon)$	Error
4 : 1	H_5	1/2	0.5091568	0.01
6 : 1	H_7	3/2	1.5079998	0.05
4 : 3	H_7	3/2	1.4478968	0.09
3 : 1	H_8	2	2.0898136	0.35

Table 1. Comparison between the analytic estimate and the numerical computation of the size of the resonance domain of four of the most prominent higher order resonances of the elastic pendulum. The second column of this table indicates the part of the expanded Hamiltonian in which the lowest order resonant terms are found.

Apart from this confirmation, we note that this splitting method preserves some geometric properties of the system such as, linear symmetry, time-reversal symmetry, the symplectic form and also the linear resonance.

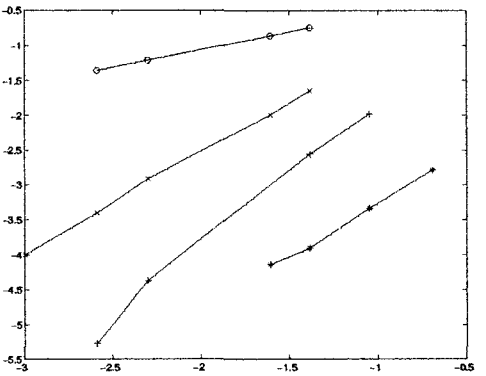

Figure 2. Plots of $\log(d_\varepsilon)$ against $\log(\varepsilon)$ for various resonances. The 4 : 1-resonance is plotted using '$-\circ$', the 3 : 1-resonance is using '$-+$', the 4 : 3-resonance is using '$-\times$' and the 6 : 1 resonance is using '$-*$'.

5 Systems with widely separated frequencies

Studies on coupled oscillator systems have a long history. They serve as models in many applications, such as vibrating mechanical structures. Most of the concern in these studies is to see how energy interchanges between the oscillators. In this frame work, our preceding sections suggest that in the higher order resonance case, the energy exchanged between the oscillators is small. This is in agreement with the traditional knowledge in this field. However, in 1990, a lot of studies (see Haller[5]) have been devoted to an extreme type of higher order resonance, i.e. systems with widely separated frequencies.

In the Hamiltonian case, Broer et.al.[3] gave a description of the unfolding of the origin of this type of system. Using normal form and singularity theory, they found that the codimension of the origin is 1 for the non semi-simple case and 3 for the semi-simple case. As a supplement to this study, we study also the dynamics in time of the semi-simple type of this system and some degeneracies due to symmetry (see Tuwankotta and Verhulst[13]).

The Hamiltonian that we consider is

$$H = \frac{1}{2}({q_1}^2 + {p_1}^2) + \frac{1}{2}\varepsilon({q_2}^2 + {p_2}^2) + H_3(\boldsymbol{q}) + H_4(\boldsymbol{q}) + \dots . \tag{8}$$

We re-scale the variables in the usual way to arrive at the perturbative Hamiltonian. The unperturbed Hamiltonian is $I_o = \frac{1}{2}({q_1}^2 + {p_1}^2)$. We then normalize (8) with respect to the S^1-action defined by the flow of the Hamiltonian vector

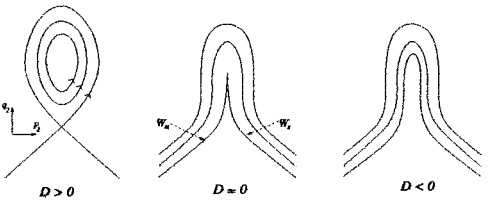

Figure 3. Phase portrait of the Poincaré section on (q_2, p_2)-plane for fixed value of $I_o \neq 0$ and $\beta \neq 0$. The geometry of phase-space of the system (9) is achieved by taking the cross product of them with S^1

field X_{I_o}. The normalized Hamiltonian is

$$H = I_o + \varepsilon \left(\frac{1}{2} \left({q_2}^2 + {p_2}^2 \right) - \alpha I_o q_2 - \frac{1}{3} \beta {q_2}^3 \right), \tag{9}$$

where α and β are parameters. If $\beta = 0$, the system degenerates to a linear system. Apart from I_o, there is one other parameter $D = 1 - 4\alpha\beta I_o$ which is important for the bifurcation scenario in this system. See Figure (3).

The conclusion of this section is that in Hamiltonian system with widely separated frequencies, there are no energy interchanges. Nevertheless, the phase interaction between the oscillators is stronger than the the one we found in the previous section. For the degenerate case and details in the study on this type of systems, see Tuwankotta and Verhulst[13].

6 Discussion

We have presented a review on higher order resonance in two degrees of freedom Hamiltonian systems. The estimate of the size of the resonance domain where the interesting dynamics takes place, has been improved. We also show numerical evidence that our new estimate is sharp. In dealing with the higher order resonance numerically, the geometric integration provides relatively cheap computation time in getting accurate results. The study of Hamiltonian systems with widely separated frequencies will be completed in the near future.

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STABILITY OF HAMILTONIAN RELATIVE EQUILIBRIA BY ENERGY METHODS

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For a compact group G a relative equilibrium of a G-invariant Hamiltonian is orbitally Liapounov stable (G-stable) if the Hessian of an augmented Hamiltonian at the corresponding critical point is definite when restricted to the symplectic normal space. This is no longer true in general for proper actions of noncompact groups, essentially because the orbit space of the coadjoint action of the group need not be Hausdorff. This paper gives a summary of our results on the stability of Hamiltonian relative equilibria. It presents a more general stability criterion that applies to relative equilibria with 'tame' angular velocities, and also gives a result that sharpens G-stability to A-stability, where A is a subset of G that depends only on the momentum of the relative equilibrium.

1 Introduction

Consider a Hamiltonian system

$$\dot{w} = J\mathrm{D}_w H(w)$$

on $M = \mathbb{R}^{2n}$ where $J = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$ and $H : M \to \mathbb{R}$ is a smooth Hamiltonian. Let 0 be an equilibrium of this system, ie $J\mathrm{D}_w H(0) = 0$. It is well-known and easily checked that the linearization $L = J\mathrm{D}_w^2 H(0)$ of the Hamiltonian system at 0 is infinitesimally symplectic, ie satisfies $JL = -L^T J$. So if λ is an eigenvalue of L then so is $-\lambda$. As a consequence, a necessary condition for the equilibrium to be stable is that $\Re\mathrm{spec}(L) = 0$. This means that nonlinear stability of an equilibrium of a Hamiltonian system can not be inferred from linear information, in contrast to the situation for general non-Hamiltonian systems. A simple sufficient condition for stability uses the conservation of energy: $H(w(t)) = H(w(0))$ **for all solutions** $w(t)$ of the Hamiltonian system.

If $\mathrm{D}^2H(0)$ is definite at 0 then 0 is a minimum or maximum of H and therefore Liapounov stable.

The aim of this paper, which summarises a longer paper[13], is to generalize this approach to relative equilibria of Hamiltonian systems with symmetry.

2 Hamiltonian systems with symmetry

Let M be a finite dimensional symplectic manifold, G a Lie group which acts properly and (for simplicity only) *freely* on M, and ω a G-invariant symplectic form on M. Let $H : M \to \mathbb{R}$ be a G-invariant Hamiltonian, and

$$\dot{x} = f_H(x) \tag{1}$$

the Hamiltonian system defined by $\omega_x(f_H(x), v) = \mathrm{D}H(x)v$ for $v \in T_xM$. Let $\mathbf{g}$ denote the Lie algebra of G and $\mathbf{g}^*$ its dual. By Noether's Theorem for each continuous symmetry $\xi \in \mathbf{g}$ locally there is a conserved quantity $\mathbf{J}(\xi)$ of (1) which is linear in ξ, so that $\mathbf{J}$ maps into $\mathbf{g}^*$ (see eg Marsden and Ratiu[7]). We will assume that $\mathbf{J}$ exists globally on M and is equivariant with respect to the usual coadjoint action of G on $\mathbf{g}^*$, given by $g\mu := (\mathrm{Ad}_g^*)^{-1}\mu$ where $\mathrm{Ad}_g : \mathbf{g} \to \mathbf{g}$ defined by $\mathrm{Ad}_g\eta = g\eta g^{-1}$ is the adjoint action. We also define the adjoint action of $\mathbf{g}$ on itself by $\mathrm{ad}_\xi\eta = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{\exp(t\xi)}\eta|_{t=0} = [\xi, \eta]$ and the dual coadjoint action of $\mathbf{g}$ on $\mathbf{g}^*$ by $\xi\mu = -\mathrm{ad}_\xi^*\mu$. A momentum value μ is defined to be *regular* if the coadjoint orbit of every nearby point has the same dimension as $G\mu$. The set of regular momenta is open and dense in $\mathbf{g}^*$ and if μ is regular then the coadjoint isotropy algebra $\mathbf{g}_\mu = \{\xi \in \mathbf{g} : \mathrm{ad}_\xi^*\mu = 0\}$ is Abelian.

3 Relative equilibria and skew product equations

A group orbit Gp is called a *relative equilibrium* if it is an equilibrium in the space of group orbits, ie if the solution of (1) with initial value p satisfies $x(t; p) \in Gp$ for all $t \in \mathbb{R}$. It is easily checked that Gp is a relative equilibrium if and only if there exists $\xi_p \in \mathbf{g}$, called the *drift velocity* of the relative equilibrium, such that $\xi_p p := \frac{\mathrm{d}}{\mathrm{d}t}\exp(t\xi_p)p|_{t=0} = f_H(p)$. Let U be a G-invariant neighbourhood of Gp in M and let $N \subset T_pM$ be a normal space to the group orbit $\mathbf{g}p$ at p, so that $\mathbf{g}p \oplus N = T_pM$. Using Palais slice coordinates[11] we can write every $x \in U$ as $x \simeq (g, v)$ where $g \in G$ and $v \in N$, see Figure 1. In these coordinates (1) takes the form

$$\dot{g} = gf_G(v), \quad \dot{v} = f_N(v). \tag{2}$$

This was proved for general systems by Krupa[3] and Fiedler et al[2]. The first equation describes the motion along group orbits, and the second equation the shape dynamics.

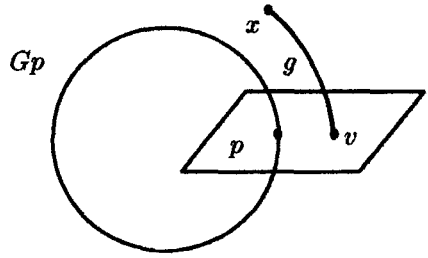

Figure 1. Palais coordinates near Gp

4 Hamiltonian relative equilibria

Examples of Hamiltonian relative equilibria include rotating molecules[9] and rotating and translating rigid bodies in fluids[4]. In the first case the effective symmetry group is compact since without loss of generality the centre of mass can be assumed to be fixed, but in the second case the symmetry group is the noncompact special Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \triangleright \mathbb{R}^3$ of rotations and translations of three space.

Stability results for relative equilibria with regular momenta have been obtained by Arnold[1] and Libermann and Marle[6], and for relative equilibria with arbitrary momenta and compact symmetry groups G (or 'almost compact' symmetry groups, see below) by Montaldi[8], Lerman and Singer[5] and Ortega and Ratiu[10]. These results do not apply to rigid bodies in fluids with non-regular momenta. To cover these Leonard and Marsden[4] presented some stability results for noncompact groups which are semidirect products of vector spaces and compact groups. We have improved these results and developed a stability theory via energy methods for general proper noncompact group actions. The key ideas are summarized below. Our main tool in this paper are the bundle equations (3) for Hamiltonian systems that we derived with Lamb[14]. We recall these in the next section.

5 Hamiltonian systems near group orbits

Let $\mu = \mathbf{J}(p)$ be the momentum of $p \in M$, let G_μ be its isotropy with respect to the coadjoint group action and $\mathbf{g}_\mu$ be its Lie algebra. In this paper we assume

that μ is *split*, ie there is a G_μ^0-invariant complement to $\mathbf{g}_\mu$ in $\mathbf{g}$, where G_μ^0 is the identity component of G_μ. The slice N to Gp at p from Section 3 has the Witt decomposition $N = N_0 \oplus N_1$. The space N_0 is isomorphic to $\mathbf{g}_\mu^*$ and can be identified with an affine subspace of $\mathbf{g}^*$ through μ which is transverse to $G\mu$ at μ. The space N_1 is a slice to $G_\mu p$ in the momentum level set $\mathbf{J}^{-1}(\mu)$. The symplectic form on T_pM restricts to a symplectic form on N_1, which we therefore call the *symplectic normal space* at p. See eg Roberts et al[14] for further details. Let J_{N_1} denote the skew-symmetric matrix that defines the symplectic form on N_1. We have the following theorem:

Theorem 1 [14] *For Hamiltonian systems and $\mu = \mathbf{J}(p)$ split, the bundle equations (2) near Gp take the form:*

$$\dot{g} = g\mathrm{D}_\nu h(\nu, w), \quad \dot{\nu} = \mathrm{ad}^*_{\mathrm{D}_\nu h(\nu,w)}\nu, \quad \dot{w} = J_{N_1}\mathrm{D}_w h(\nu, w) \tag{3}$$

where $h(\nu, w)$ is the function on the slice $N_0 \oplus N_1$ obtained by restricting the Hamiltonian H. If μ is regular then $\mathbf{g}_\mu$ is Abelian and so $\dot{\nu} \equiv 0$.

The first equation describes the motion of the body frame, the second equation the motion of the momenta in body coordinates and the last equation the shape dynamics. We refer to the $(\dot{\nu}, \dot{w})$ equations as the *slice equations*.

6 Hamiltonian relative equilibria and slice equations

An orbit Gp is a relative equilibrium if and only if p is a critical point of the Hamiltonian in the comoving frame $H_{\xi_p}(x) = H(x) - \mathbf{J}_{\xi_p}(x)$ and if and only if 0 is an equilibrium of the slice equations on $N = N_0 \oplus N_1$. For any subset $A \subset G$ we say that a relative equilibrium Gp is *A-stable* if for all x_0 close to p the trajectory $x(t; x_0)$ is close to Ap for all $t \in \mathbb{R}$. A relative equilibrium Gp is G-stable if and only if 0 is a stable equilibrium of the slice equations.

We see from Theorem 1 that for μ regular $\nu \in N_0$ can be treated as a parameter and so 0 is a stable equilibrium of the slice equations if it is a stable equilibrium of the ν-parametrised Hamiltonian system $\dot{w} = J_{N_1}\mathrm{D}_w h(\nu, w)$ on N_1. This is guaranteed if $\mathrm{D}_w^2 h(0)$ is definite, as we saw in Section 1. In this way we recover the stability results of Arnold[1] and Libermann and Marle[6].

For general μ, if all orbits of the $\dot{\nu}$-equation lie in spheres around 0 in N_0 then the dynamics on N_0 is always stable and energy methods only have to be used on N_1 to establish the stability of 0 for the slice equations. This assumption is satisfied if there is a G_μ-invariant inner product on $\mathbf{g}_\mu^*$. In this situation definiteness of $\mathrm{D}_w^2 h(0)$ again implies stability. Since $\mathrm{D}_w^2 h(0) = \mathrm{D}^2 H_{\xi_p}(p)|_{N_1}$ [13] we recover the energy-momentum method of Ortega and Ratiu[10] and Lerman and Singer[5].

7 G-stability of Hamiltonian relative equilibria

Now we will show how to obtain G-stability for Hamiltonian relative equilibria in the case of a general noncompact symmetry group G. From the equation $\dot{\nu} = \mathrm{ad}^*_{\mathrm{D}_\nu h}\nu$ of (3) we see that $\nu(t) \in G^0_\mu \nu_0$, which just rephrases the statement that the coordinates ν are the momenta in the body frame moving with velocity $\mathrm{D}_\nu h$.

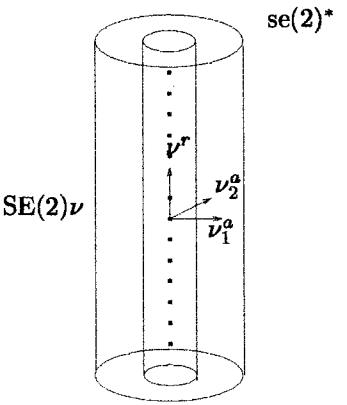

Figure 2. Coadjoint orbits for $G = \mathrm{SE}(2)$

As a motivating example consider the special Euclidean group $G = \mathrm{SE}(2) = \mathrm{SO}(2) \triangleright \mathbb{R}^2$ of the plane and $\mu = 0$. Let $\nu = (\nu^r, \nu^a)$ with $\nu^r \in \mathrm{so}(3)^*$ the angular momentum and $\nu^a = (\nu^a_1, \nu^a_2) \in (\mathbb{R}^2)^*$ the linear momentum. Then the coadjoint group orbits $G\nu$ are either cylinders with axes along the ν^r-axis, or points on the ν^r-axis, see Figure 2. We conclude that the linear momentum $\nu^a(t)$ is always bounded.

We have $\nu^a_i(t) = \nu(t)(\xi^a_i)$, $i = 1, 2$, where ξ^a_i is the i-th component of the translational part ξ^a of the velocity $\xi = (\xi^r, \xi^a) \in \mathrm{so}(2) \oplus \mathbb{R}^2$. We call the infinitesimal translations $\xi^a \in \mathfrak{t}_\mu := \mathbb{R}^2$ *tame* because $(G_\mu \nu)(\xi^a)$ is bounded for any $\nu \in \mathfrak{g}^*_\mu$, and so $\nu(t)(\xi^a)$ is bounded for any solution of the equation $\dot{\nu} = \mathrm{ad}^*_{\mathrm{D}_\nu h}\nu$. More generally, for any Lie group G and split momentum μ we call $\xi \in \mathfrak{g}_\mu$ *tame* if $G^0_\mu \xi$ is bounded. We denote the vector space of all tame velocities for μ by $\mathfrak{t}_\mu$.

In the SE(2)-example we see that $\nu^r(t) \in \mathfrak{w}^*_\mu := \mathrm{ann}_{\mathfrak{g}^*_\mu}(\mathfrak{t}_\mu)$ can grow unboundedly along the ν^r-axis of the cylinder. Here $\mathrm{ann}_{\mathfrak{g}^*_\mu}(\mathfrak{t}_\mu)$ denotes the annihilator of $\mathfrak{t}_\mu$ in $\mathfrak{g}^*_\mu$. The vector space $\mathfrak{w}^*_\mu$ defined in this way for general

groups G is called the space of *wild momenta*. We have the following theorem:

Theorem 2 [13] *The relative equilibrium Gp is G-stable if a) its drift velocity ξ_p is tame and b) $\mathrm{D}^2 H_{\xi_p}(p)|_{\mathbf{w}_\mu^* \oplus N_1}$ is definite.*

The tameness of ξ_p is necessary to get a critical point of the Hamiltonian h on $\mathbf{w}_\mu^* \oplus N_1$. If this is satisfied then definiteness of $\mathrm{D}^2 h|_{\mathbf{w}_\mu^* \oplus N_1}$ yields stability of the equilibrium 0 of the slice equations on $N_0 \oplus N_1$ and the equality $\mathrm{D}^2 h(0) = \mathrm{D}^2 H_{\xi_p}(p)|_N$ (see Patrick et al[13]) gives the result. The theorem generalizes the energy-momentum method of Lerman and Singer[5] and Ortega and Ratiu[10] to general noncompact groups. We show in Patrick et al[13] that in fact tameness of ξ_p is a *necessary* condition for any energy-momentum or energy-Casimir method to apply.

Notice that in the SE(2) example instability on $N_0 \cong \mathbf{g}_\mu^*$ is only possible along the ν^r-axis. This is the set of momenta ν which cannot be separated from 0 by open neighbourhoods in the space of group orbits $\mathbf{g}^*/G$. In other words, instability is possible exactly where $\mathbf{g}^*/G$ is not Hausdorff. This is no coincidence. Indeed, the stability theory described in Patrick et al[13] uses this as its starting point and the generalised energy-momentum criteria are deduced from topological stability results on non-Hausdorff topological spaces.

8 *A*-stability of Hamiltonian relative equlibria

In this section we show that a G-stable relative equilibrium of (1) is actually A-stable for A a certain subset of G that depends only on μ. Again for simplicity we assume that $\mu = \mathbf{J}(p)$ is split. Then by (3) the drift of nearby solutions is governed by the equation $\dot{g} = g\mathrm{D}_\nu h(\nu, w)$, and since $\mathrm{D}_\nu h(\nu, w) \in \mathbf{g}_\mu$ we have $g(t) \in g(0)G_\mu$. So the drift evolution $g(0)^{-1}g(t)$ is in the momentum isotropy subgroup G_μ^0. Since any element x close to p has the bundle coordinates $x \simeq (g, \nu, w)$ with g near id, and ν, w small we get the following theorem:

Theorem 3 [13] *Let Gp be a G-stable relative equilibrium with split momentum $\mu = \mathbf{J}(p)$. Then Gp is A-stable where $A = \bigcup_{g \in W} gG_\mu^0 g^{-1}$ for any neighbourhood W of* id $\in G$.

This result can be improved further by decomposing $G_\mu^0 = L_\mu K_\mu$ where K_μ is a subgroup for which there exists a K_μ-invariant inner product on $\mathbf{g}^*$. This can always be done using the Levi decomposition, for which K_μ is a maximal compact subgroup of G_μ^0, but some case K_μ can be chosen to be larger than this. If G_μ^0 is decomposed in this way then the subset A defined in the theorem can be replaced by $A = L_\mu^W K_\mu$ where $L_\mu^W = \bigcup_{g \in W} gL_\mu g^{-1}$ for any neighbourhood W of id $\in G$. In particular, if there exists a G_μ^0-invariant inner product on $\mathbf{g}^*$ then a G-stable relative equilibrium with momentum μ is

G_μ^0-stable, generalising results of Patrick[12], Lerman and Singer[5] and Ortega and Ratiu[10].

As an example consider a rigid body in a fluid with coincident centres of mass and buoyancy[4]. Then the symmetry group G is the special Euclidean group $G = \mathrm{SE}(3) = \mathrm{SO}(3) \triangleright \mathbb{R}^3$ of three space. If μ is regular then $G_\mu^0 = \mathrm{SO}(2) \times \mathbb{R}$. A relative equilibrium Gp with this momentum $\mu = \mathbf{J}(p)$ typically translates along and rotates around the SO(2)-axis, which we will assume is the x_1-axis. Denote rotations about the x_1-axis by $\mathrm{SO}(2)_1$ and translations along the x_1-axis by $\mathbb{R}_1$. Then $G_\mu^0 = \mathrm{SO}(2)_1 \times \mathbb{R}_1 = \mathbb{R}_1 \mathrm{SO}(2)_1$ and we can take $K_\mu = \mathrm{SO}(2)_1$ and $L_\mu = \mathbb{R}_1$. The relative equilibrium is G-stable if $\mathrm{D}^2 H_{\xi_p}(p)|_{N_1}$ is definite. If it is G-stable then by Theorem 3 and the following paragraph it is also A-stable with $A = \mathrm{SO}(2)_1 \times A^a$ where $A^a = \bigcup_{R \in W} R\mathbb{R}_1$ for any neighbourhood W of $\mathrm{id} \in \mathrm{SO}(3)$. The set A^a is a cone around the a_1-axis in the group of translations $\mathbb{R}^3$, see Figure 3. The cone can be made arbitrarily 'narrow' by choosing W sufficiently small, and trajectories will remain near the resulting set Ap by choosing the initial conditions sufficiently close to p. Leonard and Marsden[4] observed in numerical simulations that nearby solutions indeed drift within such a cone.

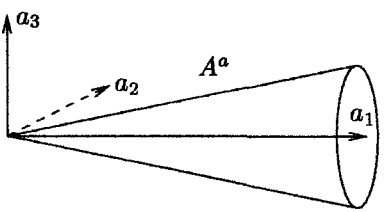

Figure 3. The cone A^a

9 Conclusion

In this paper we have given a summary of our results on the stability of Hamiltonian relative equilibria. In contrast to our forthcoming article[13] we have restricted attention to the simplest possible cases and results, hopefully getting across the main ideas at the expense of generality and optimality. In subsequent publications we will present the general theory and apply them to a variety of examples.

Acknowledgements

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TOPOLOGICALLY UNAVOIDABLE DEGENERACIES IN BAND STRUCTURE OF SOLIDS

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A new kind of topologically unavoidable branch crossing in band structure of solids is established. It is proven that there exists a great variety of 4-branch energy bands in crystalline solids of the orthorhombic system in which the crossings are necessitated by symmetry and topology. These crossings are different from conventional degeneracies that follow from space group symmetry alone.

Symmetry labeling of eigenfunctions and energies in band structure of solids was introduced in a famous paper by Bouckaert, Smoluchowski and Wigner (BSW) [1]. They used symmetry points in k-space and the irreducible representations of their symmetry groups G_k for this labeling. This approach reflects the local symmetry of the extended Bloch functions at the symmetry points and their vicinity in the Brillouin zone. As is well known Bloch functions can be built from atomic-like functions or Wannier functions [2], which are localized in configuration space and carry the full information of the Bloch functions but are not eigenfunctions of the problem. The symmetry properties of Wannier functions were dealt with by Burneika and Levinson [3], by des Cloizeaux [4], by Kovalev [5] and later by Zak [6] who defined the concept of band representations of space groups. Unlike conventional representations which give locally the symmetry in k-space, band representations carry symmetry labels of entire energy bands and enable one to have a global look at them. One distinguishes between simple energy bands with a single branch (having one Bloch function at each $\vec{k}$-vector in the Brillouin zone), and composite ones with more than one branch (two or more than one Bloch function at each $\vec{k}$). When at a given $\vec{k}$ in the Brillouin zone there are d Bloch functions ($d > 1$) with the same energy, we have a d-fold degeneracy at this $\vec{k}$. BSW [1] have carried out an analysis of degeneracies in energy bands based on irreducible representations of space groups. Following this analysis Herring [7] has raised the possibility of the appearance of accidental degeneracy which may follow from symmetry and continuity of the energy as a function of $\vec{k}$ in the Brillouin zone. The main feature of accidental degeneracy is that it can be removed by changing the potential of the crystal, while keeping the symmetry unchanged. Based on the notion of band representations, it was

recently shown that in some crystals with 4-branch composite energy bands a special kind of degeneracy exists which is a consequence of symmetry and continuity, and for which the location in k-space of the degeneracy point can be moved by changing the potential, but it cannot be removed as long as the symmetry of the crystal is kept unchanged [8].

In this talk we show the existence of topologically unavoidable branch crossings in band structure of solids. This is done for a great variety of crystals with orthorhombic symmetry. The main tool is the notion of elementary band representations [9], a classification of which was given in Ref. [10]. An additional tool are the compatibility relations [1] when applied on a global scale of entire energy bands or the so-called continuity chords (See item 2 in Ref. 6). At the center of the Brillouin zone, $\vec{k} = 0$ (the Γ-point) all irreducible representations of orthorhombic space groups are one-dimensional (because the point groups for these space groups are Abelian [1,11]). On the other hand, on the surface of the Brillouin zone there are always symmetry points at which all the irreducible representations (irreps) or co-representations of non-symmorphic orthorhombic space groups are of dimensionality 2 or higher (time-reversal included). With these facts in mind it will be shown that there are numerous orthorhombic space groups for which branch crossings are unavoidable by symmetry and topology.

The idea we are going to use for proving the rules for topologically unavoidable crossings in orthorhombic crystals is as follows: Let us start with a short description of background material. An elementary energy band which corresponds to an elementary band representation [9] has a number of branches $b(\vec{w}, \rho)$ given by the formula ($\vec{w}$ is a maximal symmetry Wyckoff position [12], and ρ specifies an irreducible representation $D^{(\vec{w},\rho)}$ of the symmetry group G_w of $\vec{w}$):

$$b(\vec{w}, \rho) = \left[\dim D^{(\vec{w},\rho)}\right] \frac{|P|}{|G_w|} \tag{1}$$

where $\left[\dim D^{(\vec{w},\rho)}\right]$ is the dimension of the irreducible representation $D^{(\vec{w},\rho)}$ of G_w and $|P|$ and $|G_w|$ are the numbers of elements of the groups P (the point group of the space group of the solid) and G_w. The label $(\vec{w}, \rho)$ specifies the band representation in question, which is an induced representation of the space group G induced from $D^{(\vec{w},\rho)}$ of G_w. Being an induced representation from a finite order subgroup G_w, the band representation $(\vec{w}, \rho)$ is infinite-dimensional and covers the infinite set of functions belonging to the corresponding energy band. By reducing the band representation $(\vec{w}, \rho)$, one can find its content in irreducible representations of the space group G at each point $\vec{k}$ in the Brillouin zone. It is, in particular, easy to find this content at

the point Γ in the center $\vec{k} = 0$ of the Brillouin zone [6]. For this, one just has to find the representations of the point group P which are induced from the representation ρ of the symmetry group G_w. For orthorhombic space groups their point groups P are all Abelian and therefore their irreducible representations will all be one-dimensional at the Γ-point. As was mentioned above, for all non-symmorphic orthorhombic space groups, on the surface of the Brillouin zone there are always symmetry points at which all the irreducible representations (including co-representations) are of dimension 2 or higher [13]. For checking whether a crossing of two branches appears we are going to follow pairs of one-dimensional representations for a given elementary energy band that initiate at the Γ-point as two separate one-dimensional representations and end up at symmetry points on the surface of the Brillouin zone that have two-dimensional representations only. In order to see how this is done let us consider as an example the orthorhombic space group # 61 (D_{2h}^{15}, Pbca). This group has 2 maximal symmetry Wyckoff positions [12]:

$$a = (000), \quad b = (00\frac{c}{2}) \tag{2}$$

For a and b the symmetry groups G_a and G_b contain the unit element E and the inversion I, and are isomorphic to the point group C_i [13]. There are 4 (2 from each Wyckoff position) four-branch elementary band representations $(\vec{w}, \rho)$ ($\vec{w} = a$ or b and $\rho = 1, 2$) for the space group Pbca according to Eq. (1), since $|P| = 8$, $|G_w| = |C_i| = 2$ and each of the 2 irreducible representations of C_i are one dimensional ($\dim D^{(\vec{w},\rho)} = 1$). The elementary band representation $(a, 1)$ induced from the trivial representation of C_i contains the four even representations of the point group D_{2h} (See Table I, irreps 1-4) at the symmetry center Γ in the Brillouin zone. On the surface of the Brillouin zone there are usually a number of symmetry points which have 2-dimensional representations only. In Table II we list those points for the orthorhombic space groups that are continuously connected by symmetry points inside the Brillouin zone (column 2 in Table II) with non-trivial symmetry elements (column 3 in Table II). It is important to point out that this possibility to follow continuously some symmetry elements along lines from the center Γ of the Brillouin zone to the surface will be used below as the main tool in establishing the branch crossings. For the example under consideration (group D_{2h}^{15}), we give in Table III the characters of the two-dimensional representations at the symmetry points on the surface of the Brillouin zone X, Y, Z A, D and H at which there are only two-dimensional irreps (the characters are taken from Ref. 13, pages 101, 102). In the upper row of Table III the point group elements of Pbca are given (in general, they appear with partial translations [12,13]). In the first column of Table

III we list the symmetry points on the surface of the Brillouin zone (see Table II for their coordinates), which have two-dimensional irreps only for the space group D_{2h}^{15}. The subscripts 1 & 2 number the different irreducible representations. In the last column of Table III we list the symmetry points in the Brillouin zone which continuously connect the center Γ with the corresponding symmetry points on the surface of the Brillouin zone in the first column. The underlined characters in Table III identify the symmetry elements of the symmetry points in the last column (see also Table II). We are now ready to discuss crossings of the branches of the elementary energy band that correspond to the elementary band representation (a,1) of the orthorhombic group Pbca. As was pointed out above the band representation (a,1) contains the irreps 1-4 of D_{2h} at the Γ-point (See Table I). In which order these representations arrange according to energy cannot be determined from symmetry considerations alone and this order depends on the specific form of the periodic potential of the crystal. However, no matter which order is chosen for these 4 irreps, it is shown in what follows that the branches in the $(a,1)$ energy band have to cross. This is seen from the fact (see Table III) that in the $\Gamma - X$ direction (including X) the character $\chi(\sigma^y) = \pm 2$, while in the $\Gamma - A$ direction including A, $\chi(\sigma^y) = 0$. This leads to unavoidable crossing in either X or A-direction. Indeed, by pairing the irreps 1-4 as in Figure 1 (1&3 and 2&4) in order to avoid crossing in the X-direction for $\chi(\sigma^y) = \pm 2$, we necessarily have crossing in the A-direction for $\chi(\sigma^y) = 0$ (in this direction there are 2 choices of pairings with each of them leading to crossing, See Figure 2a,b).

In the above analyzed example for the Pbca space group we had $\chi(\sigma^y) = \pm 2$ for the X-direction. Table III shows that in the Y-direction the character of σ^z is $\chi(\sigma^z) = \pm 2$. Such a character requires the pairing of 1& 4 and 2&3. Comparing this with Fig. 1 for the X-direction, the conclusion is that there is unavoidable crossing in one of the directions X or Y. Having chosen the order of the irreps like in Fig. 1 to have $\chi(\sigma^y) = \pm 2$ in the X-direction, we shall necessarily get a crossing the Y-direction for satisfying the requirement $\chi(\sigma^z) = \pm 2$, Fig. 3.

Consider now the directions A, D and H in Table III and notice that in the A-direction, $\chi(\sigma^y) = 0$, in the D-direction $\chi(\sigma^z) = 0$ and in the H-direction $\chi(\sigma^x) = 0$. From Table I it is seen that no pairing of the irreps 1-4 exists that would lead to the vanishing of the characters for all the 3 elements σ^x, σ^y, σ^z. This means that crossing is unavoidable in one of the directions A, D or H [14]. A possible crossing situation is shown in Fig. 4,

where we have chosen the ordering of the irreps 1-4 of D_{2h} at Γ in such a way as to avoid crossing in the A and D-directions, but then there is necessarily a crossing in the H-direction (if we had chosen the ordering of the irreps like in Fig. 1, then there would be no crossing in the D and H-directions, and the crossing would necessarily appear in the A-direction) A similar analysis with similar results can be carried out for the other 3 band representations (a,2), (b,1) and (b,2).

In conclusion, we have established branch crossings in four-branch bands in crystals with orthorhombic symmetry Pbca. One can show that similar crossings appear in a great variety of other 4-branch energy energy bounds of the orthorhombic system. A detailed list of these crossings will be presented in a separate publication. Here we would like to point out that according to our preliminary results and to Ref. [15] such crossings appear in very many different materials (about 2000 crystals) which makes the results of branch crossings of much practical interest. We would also like to point out that work on this project has started with Louis Michel [8,9,16].

Tables and Figures

$D_{2h} - mmm$	E	U^x	U^y	U^z	σ^x	σ^y	σ^z	I
1	1	1	1	1	1	1	1	1
2	1	1	-1	-1	1	-1	-1	1
3	1	-1	1	-1	-1	1	-1	1
4	1	-1	-1	1	-1	-1	1	1
5	1	1	1	1	-1	-1	-1	-1
6	1	1	-1	-1	-1	1	1	-1
7	1	-1	1	-1	1	-1	1	-1
8	1	-1	-1	1	1	1	-1	-1

Table I Irreducible representations of the point group D_{2h}. The U^x is a rotation by π around the x-axis (respectively y and z-axis), the σ^x is a reflection in the plane perpendicular to x (respectively y and z). I is the inversion.

Points on surface of BZ	points inside the the BZ	symmetry for points inside the BZ
$Z(00\frac{\pi}{c})$	$(00k_z)$	$U^z\sigma^x\sigma^y$
$X(\frac{\pi}{a}00)$	(k_x00)	$U^x\sigma^y\sigma^z$
$Y(0\frac{\pi}{b}0)$	$(0k_y0)$	$U^y\sigma^x\sigma^z$
$U(\frac{\pi}{a}0\frac{\pi}{c})$	(k_x0k_z)	σ^y
$T(0\frac{\pi}{b}\frac{\pi}{c})$	$(0k_yk_z)$	σ^x
$S(\frac{\pi}{a}\frac{\pi}{b}0)$	(k_xk_y0)	σ^z
$A(k_x0\frac{\pi}{c})$	(k_x0k_z)	σ^y
$C(k_x\frac{\pi}{b}0)$	(k_xk_y0)	σ^z
$B(0k_y\frac{\pi}{c})$	$(0k_yk_z)$	σ^x
$D(\frac{\pi}{a}k_y0)$	(k_xk_y0)	σ^z
$G(\frac{\pi}{a}0k_z)$	(k_x0k_z)	σ^y
$H(0\frac{\pi}{b}k_z)$	$(0k_yk_z)$	σ^x

Table II Symmetry points and symmetry elements on the surface and inside the Brillouin zone. For the latter their symmetry elements are given in column 3. For notations see caption for Table I.

Points on surface of the BZ	Symmetry elements and characters								points inside the BZ
	E	U^x	U^y	U^z	I	σ^x	σ^y	σ^z	
$X_{1,2}$	$\underline{2}$	$\underline{0}$	0	0	0	0	$\underline{\pm 2}$	$\underline{0}$	(k_x00)
$Y_{1,2}$	$\underline{2}$	0	$\underline{0}$	0	0	$\underline{0}$	0	$\underline{\pm 2}$	$(0k_y00)$
$Z_{1,2}$	$\underline{2}$	0	0	$\underline{0}$	0	$\underline{\pm 2}$	$\underline{0}$	0	$(00k_z)$
A	$\underline{2}$	0					$\underline{0}$	0	(k_x0k_z)
D	$\underline{2}$		0			0		$\underline{0}$	(k_xk_y0)
H	$\underline{2}$			0		$\underline{0}$	0		$(0k_yk_z)$

Table III Symmetry points on the surface of the Brillouin zone (BZ) and the characters of the corresponding irreps (taken from Ref. [13]). The underlined characters correspond to symmetry elements of the points inside the BZ.

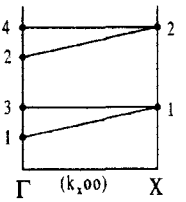

Fig. 1

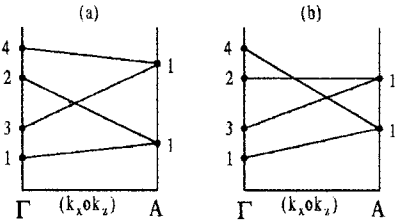

Fig. 2

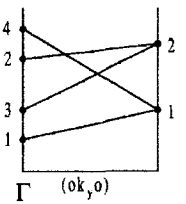

Fig. 3

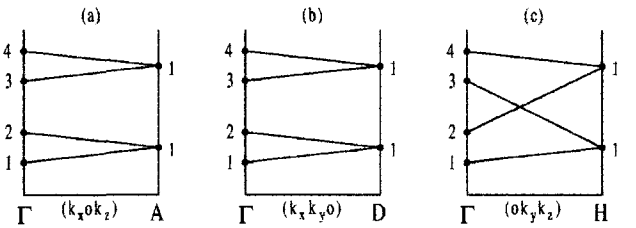

Fig. 4

Figure 1. Schematic energy graphs for the band representations (a,1) of the space group#61 in the direction X in the BZ. The numbers on the left are for the irreps of D_{2h}, and on the right for the irreps of G_X, the symmetry group of the symmetry point X.

Figure 2. 2a and 2b are schematic energy graphs like in Fig. 1, but for the A-direction. 2a and 2b give different allowed pairings or irreps.

Figure 3. Schematic energy graphs like in Fig. 1 but for the Y-direction.

Figure 4. 4a, 4b and 4c are schematic energy graphs like in Fig. 1, but for the directions A, D and H. The order of the irreps of D_{2h} was chosen in such a way that there is no crossing in A and D-directions, but then there is a necessary crossing in the H-direction.

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SYMMETRY, PERTURBATION THEORY, AND LOUIS MICHEL

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The Scientific Committee of the "Symmetry and Perturbation Theory 2001" conference has decided to organize a special session dedicated to the memory of Louis Michel, who passed away on 30 December, 1999.

Louis Michel have not participated personally to the previous "Symmetry and Perturbation Theory" meetings but his influence on the subjects encompassed by the present conference can be hardly overestimated.

Louis Michel was born in 1923 in Roanne, France, and graduated from Ecole Polytechnique. He spent a number of years in the most prestigious centers for theoretical physics, such as Niels Bohr Institute in Copenhagen (1950-1953), and The Institute of Advanced Studies in Princeton (1953-1955). In 1962, Louis was appointed as Professor of Physics at the *Institut des Hautes Etudes Scientifiques* (IHES), where he worked until his death. Louis has been a Member of the French Academy of Science since 1979 and is a recipient of the Wigner Medal (1984).

Louis started his scientific work in particle physics with publications in such a reputable journal as *Nature*. The "Michel parameter", introduced by Louis for the description of muon decay into an electron and two neutrinos, has been widely used in the interpretation of experimental data until today.

Symmetry arguments were always his main tools, starting from the discovery of isotopic parity (later known as "G-parity") and the description of polarization of elementary particles. Analysis of conservation and of violation of symmetry in different domains of physics became the favorite subject of his scientific interest.

Louis always tried to give mathematically precise formulations of symmetry arguments in physics and to apply general statements to concrete physical systems where the mathematical predictions could be verified in details.

Louis liked to state: "Symmetry enters physics through the group actions". That is why he devoted a large part of his life to the development of mathematical methods related to the study of group actions for physical problems. Whereas a lot of scientists worked on application of linear representations of groups in physics, Louis Michel studied general non-linear actions,

formulated important physical statement using such mathematical notions as group orbits, strata, group extensions, cohomology, and so on. Now we see clearly that these important notions becomes more and more appreciated in different domains of physics because they enable us to unify our models and our language.

It should be noted that quite often scientists working in different fields could hardly understand each other even when mathematical models seems to be quite similar. Louis discussed easily with representatives of both community, with mathematicians working on abstract theory and with physicists working in different fields, particle physics, solid state, crystallography, astrophysics, molecular and atomic physics, quantum chemistry.

He also was always ready to discuss concrete problems with young physicists. His reviews were addressed mainly to this new young generation of scientists with the hope that the general methods and models he developed will be applied in wider and wider fields with more and more concrete particular results.

We give below a short list of Louis publications which includes his main reviews [7,8,15,24,25,26,27,28] and some selected papers to show different subjects studied by Louis Michel in various periods of his life.

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CONFERENCE PROGRAM

Sunday 6

Arrival Day

Monday 7 – morning

Ferdinand Verhulst, *Bifurcations and limit sets in autoparametric systems*

M. Esmeralda Sousa Dias, *Pseudo-rigid bodies*

Giuseppe Marmo, *Alternative descriptions of quantum systems*

Antonio Degasperis, *Quantization of classically equivalent harmonic oscillators*

Monday 7 – afternoon

André Vanderbauwhede, *On the continuation of periodic orbits in symmetric hamiltonian systems*

Claudia Wulff, *Stability of hamiltonian relative equilibria*

Jan Sanders, *Integrable systems and Riemannian manifolds with constant curvature*

Jos Fernando Cariñena, *A geometric approach to Lie systems*

Dario Bambusi, *Some methods of normal forms in quantum mechanics*

Tuesday 8 – morning

Bob Rink, *Symmetry and resonance in periodic FPU chains*

Benoit Grebert, *Perturbations of NLS equation*

Massimiliano Berti, *A functional analysis approach to Arnold Diffusion*

Juan-Pablo Ortega, *Critical point theory and Hamiltonian dynamics around critical elements*

Nikolai Nekhoroshev, *Generalizations of Gordon's theorem*

Tuesday 8 – afternoon

Peter Olver, *Moving frames*

Maria Santos Bruzón, *On symmetries for a Boussinesq equation with nonlinear dispersion*

Monica Conti, *Oscillating solutions to fourth order superlinear differential equations*

Claudio Giberti, *Tori breakdown in coupled map lattices*

Frederic Laurent-Polz, *Point vortices on the sphere*

Faruk Gungor, *Symmetries and invariant solutions of the two dimensional Burgers equation with variable coefficients*

M. Concepcion Muriel, *Ordinary differential equations and non-solvable symmetry algebras*

Simone Paleari, *Periodic solutions for resonant nonlinear PDEs*

Wednesday 9 – morning

Ana Paula Da Silva Dias, *Heteroclinic cycles and wreath product symmetries*

Patricia Yanguas, *Computing invariant manifolds of perturbed dynamical systems*

Stella Maria Costa Abreu, *Reduction of Phase Space in Symmetric Vector Fields*

Jing Ping Wang, *Classification of integrable systems with number theoretic methods*

Gianne Derks, *The symplectic Evans matrix and solitary wave instability*

Wednesday 9 – afternoon

Excursion to Cala Luna

Thursday 10 – morning

Gianpaolo Valente, *Possible ground states of D-wave condensate in isotropic space through geometric invariant theory*

Ivan Sergienko, *Parent phase as a zero appoximation in phase transition theory*

Vladimir Tyuterev, *Perturbation theory and effective Hamiltonians in molecular spectroscopy*

Florin Diacu, *Periodic orbits of the classical atom*

Susanna Terracini, *Nehari problem in two dimensions and competing species systems*

Thursday 10 – afternoon

Genrich Belitskii, *Local normal forms of C-infinity vector fields*

Jeroen Lamb, *Normal form theory of linear reversible equivariant systems*

Juan Tomas Lazaro Ochoa, *Pseudo-normal forms and their applications*

Theo Tuwankotta, *An Extreme Type of Higher Order Resonance in Hamiltonian Systems*

Alexander Kopanskii, *Sternberg-Chen Theorem for Equivariant Hamiltonian Vector Fields*

Friday 11 – morning

Special session dedicated to L. Michel (organized by B. Zhilinskii)

- Dmitrii Sadovskii, *Normal forms, geometry, and dynamics of atomic and molecular systems with symmetry*
- Joshua Zak, *Topologically unavoidable degeneracy in band structure of solids*
- James Montaldi, *Group theoretic criteria for robust homoclinic connections*
- Yuri Gufan, *Discrepancies between the concepts of structural stability in mathematics and phase transitions theory*

Friday 11 – afternoon Informal session on normal forms, coordinated by Sebastian Walcher.

Hans-Peter Kruse, *A two-dimensional version of the Camassa-Holm equation*

Simonetta Abenda, *Geometry and dynamics of algebraically integrable systems with deficiency: the case of hyperelliptically separable systems*

Maria Luz Gandarias, *Symmetry analysis and reductions of the Schwarz-KdV equation in 2+1 dimensions*

Saturday 12 – morning

Faridon Amdjadi, *Symmetry breaking Hopf bifurcations in problems with O(2) symmetry*

Vladimir Rosenhaus, *One-dimensional infinite symmetries, boundary conditions and local conservation laws*

Tudor Ratiu, *A review of Lagrangian reduction*

Mark Roberts, *Bifurcations of Hamiltonian relative equilibria*

Sunday 13

Departure Day

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These papers appeared in preliminary form on the SPT website, *http://web.tiscalinet.it/SPT2001/spt.html*; they will appear as a special issue of the Journal *Acta Applicandae Mathematicae.*

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